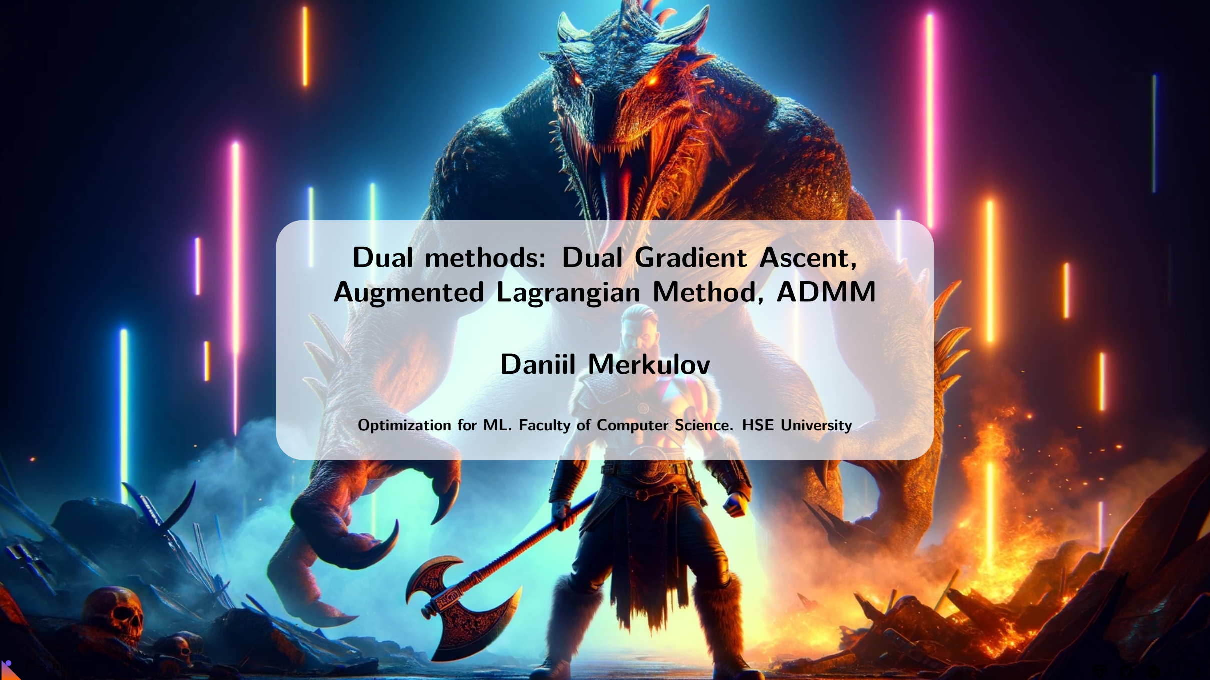


Dual methods: Dual Gradient Ascent, Augmented Lagrangian Method, ADMM

Daniil Merkulov

Optimization for ML. Faculty of Computer Science. HSE University

Why do we want to solve dual problems?

Primal problem

$$
\begin{aligned}
& f_0(x) \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } \quad & f_i(x) \leq 0, \; i = 1, \ldots, m \\
& h_i(x) = 0, \; i = 1, \ldots, p
\end{aligned}
$$

Dual problem

$$
\begin{aligned}
& g(\lambda, \nu) = \min_{x \in \mathcal{D}} L(x, \lambda, \nu) = \\
& \min_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x) \right) \quad \to \max_{\lambda \in \mathbb{R}^m, \nu \in \mathbb{R}^p} \\
& \text{s.t. } \lambda \succeq 0
\end{aligned}
$$

- **Shadow Prices.** In economics and resource allocation problems, dual variables can be interpreted as shadow prices, providing economic insights into resource utilization and constraints.

Why do we want to solve dual problems?

Primal problem

$$
\begin{split}
& f_0(x) \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } \quad & f_i(x) \leq 0, \; i = 1, \ldots, m \\
& h_i(x) = 0, \; i = 1, \ldots, p
\end{split}
$$

Dual problem

$$
\begin{split}
g(\lambda, \nu) = \min_{x \in \mathcal{D}} L(x, \lambda, \nu) = \\
\min_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x) \right) \quad \to \max_{\lambda \in \mathbb{R}^m, \nu \in \mathbb{R}^p} \\
\text{s.t. } \lambda \succeq 0
\end{split}
$$

- **Shadow Prices.** In economics and resource allocation problems, dual variables can be interpreted as shadow prices, providing economic insights into resource utilization and constraints.
- **Market Equilibrium.** Dual problems often represent market equilibrium conditions, making them essential for economic modeling and analysis.

Why do we want to solve dual problems?

Primal problem

$$
\begin{aligned}
& f_0(x) \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } \quad & f_i(x) \leq 0, \; i = 1, \ldots, m \\
& h_i(x) = 0, \; i = 1, \ldots, p
\end{aligned}
$$

Dual problem

$$
g(\lambda, \nu) = \min_{x \in \mathcal{D}} L(x, \lambda, \nu) =
$$

$$
\min_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x) \right) \quad \to \max_{\lambda \in \mathbb{R}^m, \nu \in \mathbb{R}^p}
$$

$$
\text{s.t. } \; \lambda \succeq 0
$$

- **Shadow Prices.** In economics and resource allocation problems, dual variables can be interpreted as shadow prices, providing economic insights into resource utilization and constraints.
- **Market Equilibrium.** Dual problems often represent market equilibrium conditions, making them essential for economic modeling and analysis.
- **Dual Problems Provide Bounds.** Dual problems often offer bounds on the optimal value of the primal problem. This can be useful for assessing the quality of approximate solutions.

Why do we want to solve dual problems?

Primal problem

$$
\begin{aligned}
& f_0(x) \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } \quad & f_i(x) \leq 0, \; i = 1, \ldots, m \\
& h_i(x) = 0, \; i = 1, \ldots, p
\end{aligned}
$$

Dual problem

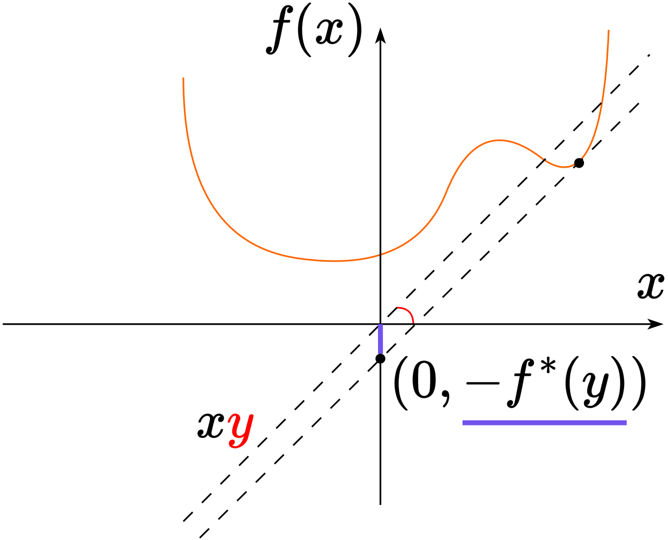
$$
g(\lambda, \nu) = \min_{x \in \mathcal{D}} L(x, \lambda, \nu) =
$$

$$
\min_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right) \quad \to \max_{\lambda \in \mathbb{R}^m, \nu \in \mathbb{R}^p}
$$

$$
\text{s.t. } \lambda \succeq 0
$$

- **Shadow Prices.** In economics and resource allocation problems, dual variables can be interpreted as shadow prices, providing economic insights into resource utilization and constraints.
- **Market Equilibrium.** Dual problems often represent market equilibrium conditions, making them essential for economic modeling and analysis.
- **Dual Problems Provide Bounds.** Dual problems often offer bounds on the optimal value of the primal problem. This can be useful for assessing the quality of approximate solutions.
- **Duality Gap.** The difference between the primal and dual solutions (duality gap) provides valuable information about the solution's optimality.

Conjugate functions

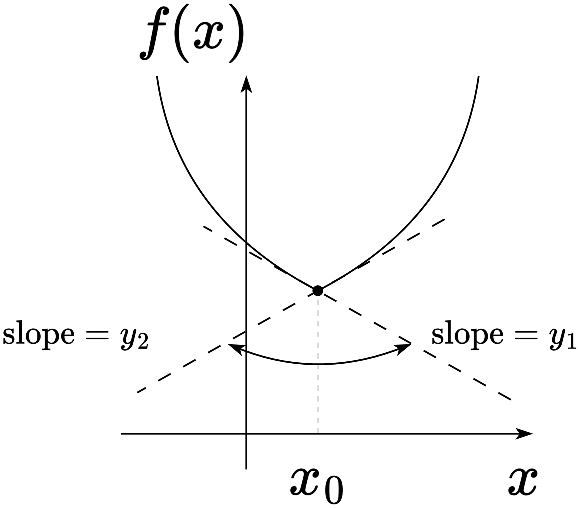


Recall that given $f : \mathbb{R}^n \to \mathbb{R}$, the function defined by

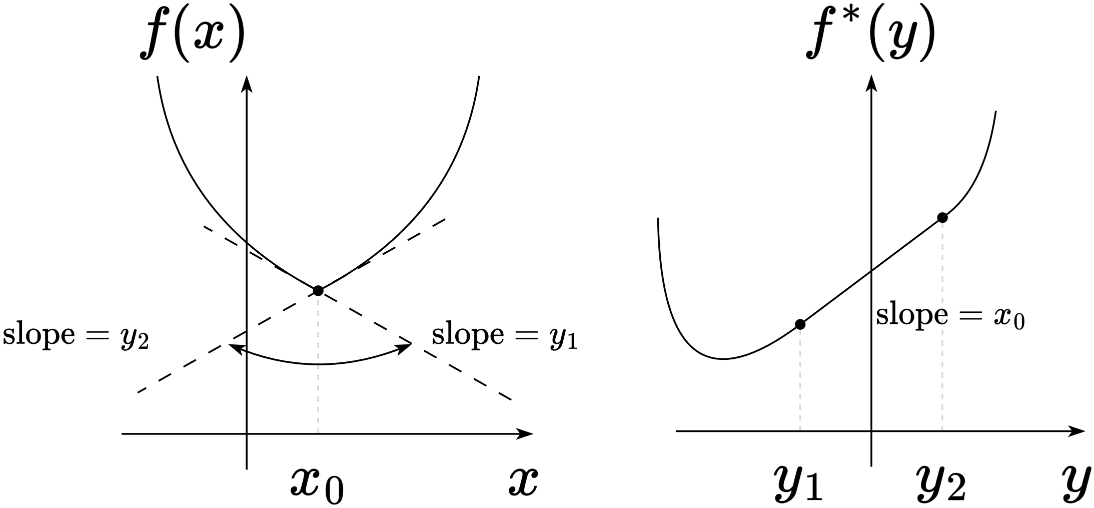
$$f^*(y) = \max_x \left[y^T x - f(x)\right]$$

is called its conjugate.

Geometrical intution



Geometrical intution



Conjugate function properties

Recall that given $f : \mathbb{R}^n \to \mathbb{R}$, the function defined by

$$f^*(y) = \max_x \left[y^T x - f(x) \right]$$

is called its conjugate.

- Conjugates appear frequently in dual programs, since

$$-f^*(y) = \min_x \left[f(x) - y^T x \right]$$

Conjugate function properties

Recall that given $f : \mathbb{R}^n \to \mathbb{R}$, the function defined by

$$f^*(y) = \max_x \left[y^T x - f(x) \right]$$

is called its conjugate.

- Conjugates appear frequently in dual programs, since

$$-f^*(y) = \min_x \left[f(x) - y^T x \right]$$

- If f is closed and convex, then $f^{**} = f$. Also,

$$x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x) \Leftrightarrow x \in \arg\min_z \left[f(z) - y^T z \right]$$

Conjugate function properties

Recall that given $f : \mathbb{R}^n \to \mathbb{R}$, the function defined by

$$f^*(y) = \max_x \left[y^T x - f(x) \right]$$

is called its conjugate.

- Conjugates appear frequently in dual programs, since

$$-f^*(y) = \min_x \left[f(x) - y^T x \right]$$

- If f is closed and convex, then $f^{**} = f$. Also,

$$x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x) \Leftrightarrow x \in \arg\min_z \left[f(z) - y^T z \right]$$

- If f is strictly convex, then

$$\nabla f^*(y) = \arg\min_z \left[f(z) - y^T z \right]$$

Conjugate function properties (proofs)

We will show that $x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x)$, assuming that f is convex and closed.

- **Proof of** $\Leftarrow$: Suppose $y \in \partial f(x)$. Then $x \in M_y$, the set of maximizers of $y^T z - f(z)$ over z. But

$$f^*(y) = \max_z \{y^T z - f(z)\} \quad \text{and} \quad \partial f^*(y) = \text{cl}(\text{conv}(\bigcup_{z \in M_y} \{z\})).$$

Thus $x \in \partial f^*(y)$.

Conjugate function properties (proofs)

We will show that $x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x)$, assuming that f is convex and closed.

- **Proof of** $\Leftarrow$: Suppose $y \in \partial f(x)$. Then $x \in M_y$, the set of maximizers of $y^T z - f(z)$ over z. But

$$f^*(y) = \max_z \{y^T z - f(z)\} \quad \text{and} \quad \partial f^*(y) = \text{cl}(\text{conv}(\bigcup_{z \in M_y} \{z\})).$$

  Thus $x \in \partial f^*(y)$.

- **Proof of** $\Rightarrow$: From what we showed above, if $x \in \partial f^*(y)$, then $y \in \partial f^*(x)$, but $f^{**} = f$.

Conjugate function properties (proofs)

We will show that $x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x)$, assuming that f is convex and closed.

- **Proof of** $\Leftarrow$: Suppose $y \in \partial f(x)$. Then $x \in M_y$, the set of maximizers of $y^T z - f(z)$ over z. But

$$f^*(y) = \max_z \{y^T z - f(z)\} \quad \text{and} \quad \partial f^*(y) = \text{cl}(\text{conv}(\bigcup_{z \in M_y} \{z\})).$$

  Thus $x \in \partial f^*(y)$.

- **Proof of** $\Rightarrow$: From what we showed above, if $x \in \partial f^*(y)$, then $y \in \partial f^*(x)$, but $f^{**} = f$.

Conjugate function properties (proofs)

We will show that $x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x)$, assuming that f is convex and closed.

- **Proof of** $\Leftarrow$: Suppose $y \in \partial f(x)$. Then $x \in M_y$, the set of maximizers of $y^T z - f(z)$ over z. But

$$
f^*(y) = \max_z \{y^T z - f(z)\} \quad \text{and} \quad \partial f^*(y) = \text{cl}(\text{conv}(\bigcup_{z \in M_y} \{z\})).
$$

  Thus $x \in \partial f^*(y)$.

- **Proof of** $\Rightarrow$: From what we showed above, if $x \in \partial f^*(y)$, then $y \in \partial f^*(x)$, but $f^{**} = f$.

Clearly $y \in \partial f(x) \Leftrightarrow x \in \arg\min_z \{f(z) - y^T z\}$

Lastly, if f is strictly convex, then we know that $f(z) - y^T z$ has a unique minimizer over z, and this must be $\nabla f^*(y)$.

Dual (sub)gradient method

Even if we can't derive dual (conjugate) in closed form, we can still use dual-based gradient or subgradient methods.

Consider the problem:

$$\min_x \quad f(x) \quad \text{subject to} \quad Ax = b$$

Dual (sub)gradient method

Even if we can't derive dual (conjugate) in closed form, we can still use dual-based gradient or subgradient methods.

Consider the problem:

$$\min_x \quad f(x) \quad \text{subject to} \quad Ax = b$$

Its dual problem is:

$$\max_u \quad -f^*(-A^T u) - b^T u$$

where f^* is the conjugate of f. Defining $g(u) = -f^*(-A^T u) - b^T u$, note that:

$$\partial g(u) = A\partial f^*(-A^T u) - b$$

Dual (sub)gradient method

Even if we can't derive dual (conjugate) in closed form, we can still use dual-based gradient or subgradient methods.

Consider the problem:

$$\min_x \quad f(x) \quad \text{subject to} \quad Ax = b$$

Its dual problem is:

$$\max_u \quad -f^*(-A^T u) - b^T u$$

where f^* is the conjugate of f. Defining $g(u) = -f^*(-A^T u) - b^T u$, note that:

$$\partial g(u) = A \partial f^*(-A^T u) - b$$

Therefore, using what we know about conjugates

$$\partial g(u) = Ax - b \quad \text{where} \quad x \in \arg\min_z \left[f(z) + u^T A z\right]$$

Dual (sub)gradient method

Even if we can't derive dual (conjugate) in closed form, we can still use dual-based gradient or subgradient methods.

Consider the problem:

$$\min_x \quad f(x) \quad \text{subject to} \quad Ax = b$$

Its dual problem is:

$$\max_u \quad -f^*(-A^T u) - b^T u$$

where f^* is the conjugate of f. Defining $g(u) = -f^*(-A^T u) - b^T u$, note that:

$$\partial g(u) = A\partial f^*(-A^T u) - b$$

Therefore, using what we know about conjugates

$$\partial g(u) = Ax - b \quad \text{where} \quad x \in \arg\min_z \left[f(z) + u^T Az\right]$$

Dual ascent method for maximizing dual objective:

$$x_k \in \arg\min_x \left[f(x) + (u_{k-1})^T Ax\right]$$

$$u_k = u_{k-1} + \alpha_k (Ax_k - b)$$

- Step sizes α_k, $k = 1, 2, 3, \ldots$, are chosen in standard ways.

Dual (sub)gradient method

Even if we can't derive dual (conjugate) in closed form, we can still use dual-based gradient or subgradient methods.

Consider the problem:

$$\min_x \quad f(x) \quad \text{subject to} \quad Ax = b$$

Its dual problem is:

$$\max_u \quad -f^*(-A^T u) - b^T u$$

where f^* is the conjugate of f. Defining $g(u) = -f^*(-A^T u) - b^T u$, note that:

$$\partial g(u) = A \partial f^*(-A^T u) - b$$

Therefore, using what we know about conjugates

$$\partial g(u) = Ax - b \quad \text{where} \quad x \in \arg\min_z \left[f(z) + u^T A z \right]$$

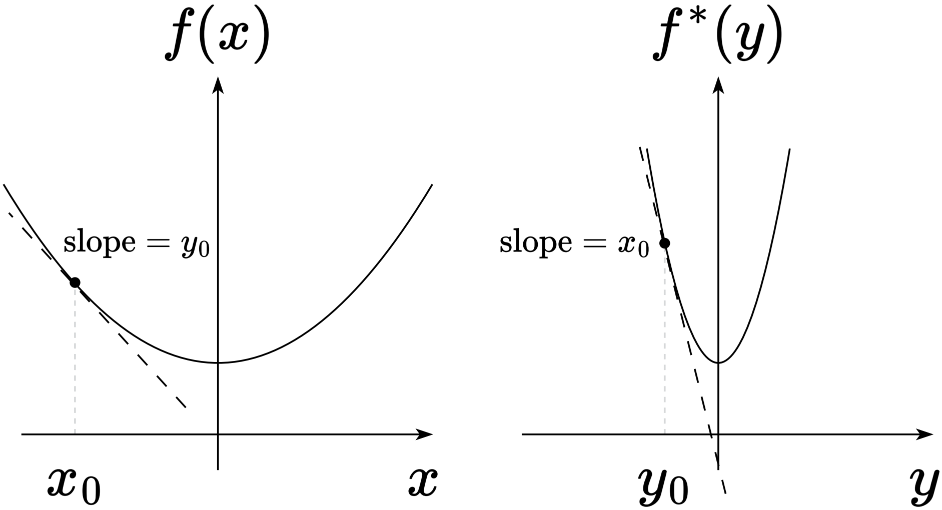
Dual ascent method for maximizing dual objective:

$$x_k \in \arg\min_x \left[f(x) + (u_{k-1})^T A x \right]$$

$$u_k = u_{k-1} + \alpha_k (A x_k - b)$$

- Step sizes α_k, $k = 1, 2, 3, \ldots$, are chosen in standard ways.
- Proximal gradients and acceleration can be applied as they would usually.

Slopes of f and f^*



Slopes of f and f^*

Assume that f is a closed and convex function. Then f is strongly convex with parameter $\mu \Leftrightarrow \nabla f^*$ is Lipschitz with parameter $1/\mu$.

Slopes of f and f^*

Assume that f is a closed and convex function. Then f is strongly convex with parameter $\mu \Leftrightarrow \nabla f^*$ is Lipschitz with parameter $1/\mu$.

Proof of "$\Rightarrow$": Recall, if g is strongly convex with minimizer x, then

$$
g(y) \geq g(x) + \frac{\mu}{2}\|y - x\|^2, \quad \text{for all } y
$$

Slopes of f and f^*

Assume that f is a closed and convex function. Then f is strongly convex with parameter $\mu \Leftrightarrow \nabla f^*$ is Lipschitz with parameter $1/\mu$.

Proof of "$\Rightarrow$": Recall, if g is strongly convex with minimizer x, then

$$g(y) \geq g(x) + \frac{\mu}{2}\|y - x\|^2, \quad \text{for all } y$$

Hence, defining $x_u = \nabla f^*(u)$ and $x_v = \nabla f^*(v)$,

$$\begin{aligned} f(x_v) - u^T x_v &\geq f(x_u) - u^T x_u + \frac{\mu}{2}\|x_u - x_v\|^2 \\ f(x_u) - v^T x_u &\geq f(x_v) - v^T x_v + \frac{\mu}{2}\|x_u - x_v\|^2 \end{aligned}$$

Slopes of f and f^*

Assume that f is a closed and convex function. Then f is strongly convex with parameter $\mu \Leftrightarrow \nabla f^*$ is Lipschitz with parameter $1/\mu$.

Proof of "$\Rightarrow$": Recall, if g is strongly convex with minimizer x, then

$$g(y) \geq g(x) + \frac{\mu}{2}\|y - x\|^2, \quad \text{for all } y$$

Hence, defining $x_u = \nabla f^*(u)$ and $x_v = \nabla f^*(v)$,

$$f(x_v) - u^T x_v \geq f(x_u) - u^T x_u + \frac{\mu}{2}\|x_u - x_v\|^2$$
$$f(x_u) - v^T x_u \geq f(x_v) - v^T x_v + \frac{\mu}{2}\|x_u - x_v\|^2$$

Adding these together, using the Cauchy-Schwarz inequality, and rearranging shows that

$$\|x_u - x_v\|^2 \leq \frac{1}{\mu}\|u - v\|^2$$

Slopes of f and f^*

Proof of "$\Leftarrow$": for simplicity, call $g = f^*$ and $L = \frac{1}{\mu}$. As ∇g is Lipschitz with constant L, so is $g_x(z) = g(z) - \nabla g(x)^T z$, hence

$$g_x(z) \leq g_x(y) + \nabla g_x(y)^T (z - y) + \frac{L}{2} \|z - y\|_2^2$$

Slopes of f and f^*

Proof of "$\Leftarrow$": for simplicity, call $g = f^*$ and $L = \frac{1}{\mu}$. As ∇g is Lipschitz with constant L, so is $g_x(z) = g(z) - \nabla g(x)^T z$, hence

$$g_x(z) \leq g_x(y) + \nabla g_x(y)^T (z - y) + \frac{L}{2}\|z - y\|_2^2$$

Minimizing each side over z, and rearranging, gives

$$\frac{1}{2L}\|\nabla g(x) - \nabla g(y)\|^2 \leq g(y) - g(x) + \nabla g(x)^T (x - y)$$

Slopes of f and f^*

Proof of "$\Leftarrow$": for simplicity, call $g = f^*$ and $L = \frac{1}{\mu}$. As ∇g is Lipschitz with constant L, so is $g_x(z) = g(z) - \nabla g(x)^T z$, hence

$$g_x(z) \leq g_x(y) + \nabla g_x(y)^T (z - y) + \frac{L}{2} \|z - y\|_2^2$$

Minimizing each side over z, and rearranging, gives

$$\frac{1}{2L} \|\nabla g(x) - \nabla g(y)\|^2 \leq g(y) - g(x) + \nabla g(x)^T (x - y)$$

Exchanging roles of x, y, and adding together, gives

$$\frac{1}{L} \|\nabla g(x) - \nabla g(y)\|^2 \leq (\nabla g(x) - \nabla g(y))^T (x - y)$$

Slopes of f and f^*

Proof of "$\Leftarrow$": for simplicity, call $g = f^*$ and $L = \frac{1}{\mu}$. As ∇g is Lipschitz with constant L, so is $g_x(z) = g(z) - \nabla g(x)^T z$, hence

$$g_x(z) \leq g_x(y) + \nabla g_x(y)^T (z - y) + \frac{L}{2}\|z - y\|_2^2$$

Minimizing each side over z, and rearranging, gives

$$\frac{1}{2L}\|\nabla g(x) - \nabla g(y)\|^2 \leq g(y) - g(x) + \nabla g(x)^T (x - y)$$

Exchanging roles of x, y, and adding together, gives

$$\frac{1}{L}\|\nabla g(x) - \nabla g(y)\|^2 \leq (\nabla g(x) - \nabla g(y))^T (x - y)$$

Let $u = \nabla f(x)$, $v = \nabla g(y)$; then $x \in \partial g^*(u)$, $y \in \partial g^*(v)$, and the above reads $(x - y)^T (u - v) \geq \frac{\|u-v\|^2}{L}$, implying the result.

Convergence guarantees

The following results hold from combining the last fact with what we already know about gradient descent: (This is ignoring the role of A, and thus reflects the case when the singular values of A are all close to 1. To be more precise, the step sizes here should be: $\frac{\mu}{\sigma_{\max}(A)^2}$ (first case) and $\frac{2}{\frac{\sigma_{\max}(A)^2}{\mu} + \frac{\sigma_{\min}(A)^2}{L}}$ (second case).)

- If f is strongly convex with parameter μ, then dual gradient ascent with constant step sizes $\alpha_k = \mu$ converges at sublinear rate $O(\frac{1}{\epsilon})$.

Convergence guarantees

The following results hold from combining the last fact with what we already know about gradient descent: (This is ignoring the role of A, and thus reflects the case when the singular values of A are all close to 1. To be more precise, the step sizes here should be: $\frac{\mu}{\sigma_{\max}(A)^2}$ (first case) and $\frac{2}{\frac{\sigma_{\max}(A)^2}{\mu}+\frac{\sigma_{\min}(A)^2}{L}}$ (second case).)

- If f is strongly convex with parameter μ, then dual gradient ascent with constant step sizes $\alpha_k = \mu$ converges at sublinear rate $O(\frac{1}{\epsilon})$.
- If f is strongly convex with parameter μ and ∇f is Lipschitz with parameter L, then dual gradient ascent with step sizes $\alpha_k = \frac{2}{\frac{1}{\mu}+\frac{1}{L}}$ converges at linear rate $O(\log(\frac{1}{\epsilon}))$.

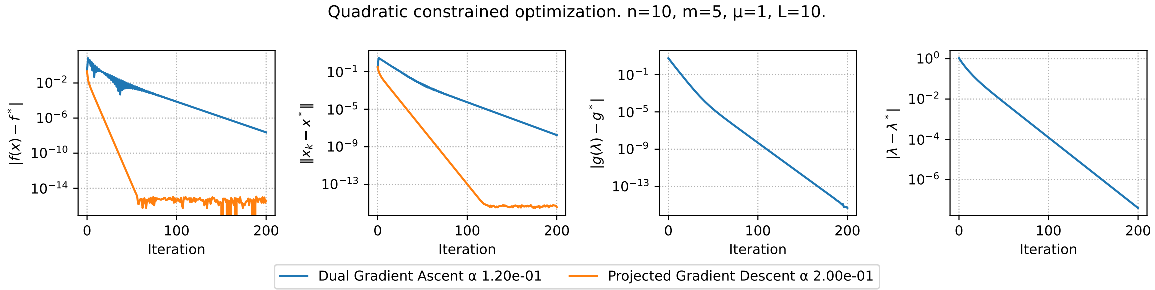
Convergence guarantees

The following results hold from combining the last fact with what we already know about gradient descent: (This is ignoring the role of A, and thus reflects the case when the singular values of A are all close to 1. To be more precise, the step sizes here should be: $\frac{\mu}{\sigma_{\max}(A)^2}$ (first case) and $\frac{2}{\frac{\sigma_{\max}(A)^2}{\mu} + \frac{\sigma_{\min}(A)^2}{L}}$ (second case).)

- If f is strongly convex with parameter μ, then dual gradient ascent with constant step sizes $\alpha_k = \mu$ converges at sublinear rate $O(\frac{1}{\epsilon})$.
- If f is strongly convex with parameter μ and ∇f is Lipschitz with parameter L, then dual gradient ascent with step sizes $\alpha_k = \frac{2}{\frac{1}{\mu} + \frac{1}{L}}$ converges at linear rate $O(\log(\frac{1}{\epsilon}))$.
- Note that this describes convergence in the dual. Convergence in the primal requires more assumptions

Example: equality constrained quadratic minimization.

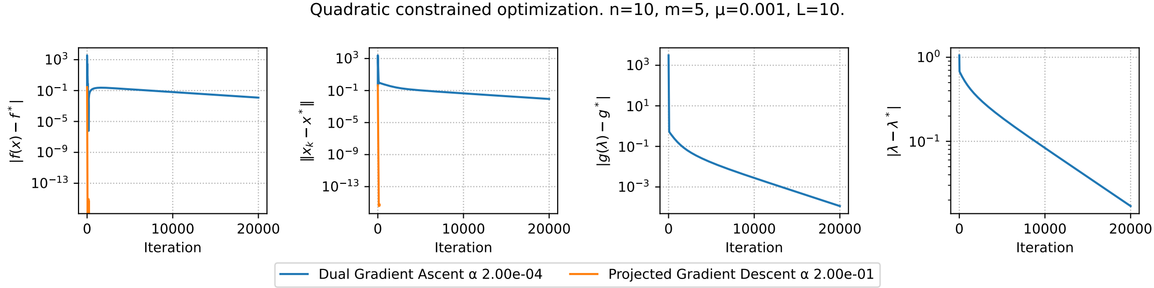
$$
f(x) = \frac{1}{2}x^TAx - b^Tx \to \min_{x \in \mathbb{R}^n} \qquad \text{subject to} \quad Cx = d, \qquad A \in \mathbb{S}^n_+, C \in \mathbb{R}^{m \times n}, m < n.
$$



We need to find a minimum of a quadratic function in some linear subspace, defined by the solution of linear equation $Cx = d$. This is a conditional optimization problem, we start from strongly convex setting.

Example: equality constrained quadratic minimization.

$$
f(x) = \frac{1}{2}x^T A x - b^T x \to \min_{x \in \mathbb{R}^n} \qquad \text{subject to} \quad Cx = d, \qquad A \in \mathbb{S}^n_+, C \in \mathbb{R}^{m \times n}, m < n.
$$



Situation is getting worse as soon as we loose strong convexity, the dual convergence will still be linear, but the rate is very low.

Dual decomposition

Consider

$$\min_x \sum_{i=1}^{B} f_i(x_i) \quad \text{subject to} \quad Ax = b$$

Dual decomposition

Consider

$$\min_x \sum_{i=1}^B f_i(x_i) \quad \text{subject to} \quad Ax = b$$

Here $x = (x_1, \dots, x_B) \in \mathbb{R}^n$ divides into B blocks of variables, with each $x_i \in \mathbb{R}^{n_i}$. We can also partition A accordingly:

$$A = [A_1 \dots A_B], \text{ where } A_i \in \mathbb{R}^{m \times n_i}$$

Dual decomposition

Consider

$$\min_x \sum_{i=1}^B f_i(x_i) \quad \text{subject to} \quad Ax = b$$

Here $x = (x_1, \dots, x_B) \in \mathbb{R}^n$ divides into B blocks of variables, with each $x_i \in \mathbb{R}^{n_i}$. We can also partition A accordingly:

$$A = [A_1 \dots A_B], \text{ where } A_i \in \mathbb{R}^{m \times n_i}$$

Simple but powerful observation, in calculation of subgradient, is that the minimization decomposes into B separate problems:

$$x^{\text{new}} \in \arg\min_x \left(\sum_{i=1}^B f_i(x_i) + u^T A x \right)$$

$$\Rightarrow x_i^{\text{new}} \in \arg\min_{x_i} \left(f_i(x_i) + u^T A_i x_i \right), \quad i = 1, \dots, B$$

$$x_i^k \in \arg\min_{x_i} \left(f_i(x_i) + (u^{k-1})^T A_i x_i \right), \quad i = 1, \dots, B$$

$$u^k = u^{k-1} + \alpha_k \left(\sum_{i=1}^B A_i x_i^k - b \right)$$

Dual decomposition

Consider

$$\min_x \sum_{i=1}^B f_i(x_i) \quad \text{subject to} \quad Ax = b$$

Here $x = (x_1, \ldots, x_B) \in \mathbb{R}^n$ divides into B blocks of variables, with each $x_i \in \mathbb{R}^{n_i}$. We can also partition A accordingly:

$$A = [A_1 \ldots A_B], \text{ where } A_i \in \mathbb{R}^{m \times n_i}$$

Simple but powerful observation, in calculation of subgradient, is that the minimization decomposes into B separate problems:

$$x^{\text{new}} \in \arg\min_x \left(\sum_{i=1}^B f_i(x_i) + u^T A x \right)$$

$$\Rightarrow x_i^{\text{new}} \in \arg\min_{x_i} \left(f_i(x_i) + u^T A_i x_i \right), \quad i = 1, \ldots, B$$

$$x_i^k \in \arg\min_{x_i} \left(f_i(x_i) + (u^{k-1})^T A_i x_i \right), \quad i = 1, \ldots, B$$

$$u^k = u^{k-1} + \alpha_k \left(\sum_{i=1}^B A_i x_i^k - b \right)$$

Can think of these steps as:

- **Broadcast:** Send u to each of the B processors, each optimizes in parallel to find x_i.

Dual decomposition

Consider

$$\min_x \sum_{i=1}^B f_i(x_i) \quad \text{subject to} \quad Ax = b$$

Here $x = (x_1, \dots, x_B) \in \mathbb{R}^n$ divides into B blocks of variables, with each $x_i \in \mathbb{R}^{n_i}$. We can also partition A accordingly:

$$A = [A_1 \dots A_B], \text{ where } A_i \in \mathbb{R}^{m \times n_i}$$

Simple but powerful observation, in calculation of subgradient, is that the minimization decomposes into B separate problems:

$$x^{\text{new}} \in \arg\min_x \left(\sum_{i=1}^B f_i(x_i) + u^T Ax \right)$$

$$\Rightarrow x_i^{\text{new}} \in \arg\min_{x_i} \left(f_i(x_i) + u^T A_i x_i \right), \quad i = 1, \dots, B$$

$$x_i^k \in \arg\min_{x_i} \left(f_i(x_i) + (u^{k-1})^T A_i x_i \right), \quad i = 1, \dots, B$$

$$u^k = u^{k-1} + \alpha_k \left(\sum_{i=1}^B A_i x_i^k - b \right)$$

Can think of these steps as:

- **Broadcast:** Send u to each of the B processors, each optimizes in parallel to find x_i.
- **Gather:** Collect $A_i x_i$ from each processor, update the global dual variable u.

Inequality constraints

Consider the optimization problem:

$$\min_x \sum_{i=1}^B f_i(x_i) \quad \text{subject to} \quad \sum_{i=1}^B A_i x_i \leq b$$

Inequality constraints

Consider the optimization problem:

$$\min_x \sum_{i=1}^B f_i(x_i) \quad \text{subject to} \quad \sum_{i=1}^B A_i x_i \leq b$$

Using **dual decomposition**, specifically the **projected subgradient method**, the iterative steps can be expressed as:

- The primal update step:

$$x_i^k \in \arg\min_{x_i} \left[f_i(x_i) + \left(u^{k-1}\right)^T A_i x_i \right], \quad i = 1, \ldots, B$$

Inequality constraints

Consider the optimization problem:

$$\min_x \sum_{i=1}^B f_i(x_i) \quad \text{subject to} \quad \sum_{i=1}^B A_i x_i \leq b$$

Using **dual decomposition**, specifically the **projected subgradient method**, the iterative steps can be expressed as:

- The primal update step:

$$x_i^k \in \arg\min_{x_i} \left[f_i(x_i) + \left(u^{k-1}\right)^T A_i x_i \right], \quad i = 1, \dots, B$$

- The dual update step:

$$u^k = \left(u^{k-1} + \alpha_k \left(\sum_{i=1}^B A_i x_i^k - b \right) \right)_+$$

where $(u)_+$ denotes the positive part of u, i.e., $(u_+)_i = \max\{0, u_i\}$, for $i = 1, \dots, m$.

Price Coordination Interpretation (Vandenberghe)

- **System Overview**: Consider a system with B units, where each unit independently chooses its decision variable x_i, which determines how to allocate its goods.

Price Coordination Interpretation (Vandenberghe)

- **System Overview**: Consider a system with B units, where each unit independently chooses its decision variable x_i, which determines how to allocate its goods.
- **Resource Constraints**: These are limits on shared resources, represented by the rows of A. Each component of the dual variable u_j represents the price of resource j.

Price Coordination Interpretation (Vandenberghe)

- **System Overview**: Consider a system with B units, where each unit independently chooses its decision variable x_i, which determines how to allocate its goods.
- **Resource Constraints**: These are limits on shared resources, represented by the rows of A. Each component of the dual variable u_j represents the price of resource j.
- **Dual Update Rule**:

$$u_j^{\text{new}} = (u_j - ts_j)_+, \quad j = 1, \ldots, m$$

where $s = b - \sum_{i=1}^{B} A_i x_i$ represents the slacks.

Price Coordination Interpretation (Vandenberghe)

- **System Overview**: Consider a system with B units, where each unit independently chooses its decision variable x_i, which determines how to allocate its goods.
- **Resource Constraints**: These are limits on shared resources, represented by the rows of A. Each component of the dual variable u_j represents the price of resource j.
- **Dual Update Rule**:

$$u_j^{\text{new}} = (u_j - ts_j)_+, \quad j = 1, \dots, m$$

  where $s = b - \sum_{i=1}^{B} A_i x_i$ represents the slacks.
- **Price Adjustments**:

Price Coordination Interpretation (Vandenberghe)

- **System Overview**: Consider a system with B units, where each unit independently chooses its decision variable x_i, which determines how to allocate its goods.
- **Resource Constraints**: These are limits on shared resources, represented by the rows of A. Each component of the dual variable u_j represents the price of resource j.
- **Dual Update Rule**:

$$u_j^{\text{new}} = (u_j - ts_j)_+, \quad j = 1, \dots, m$$

  where $s = b - \sum_{i=1}^{B} A_i x_i$ represents the slacks.
- **Price Adjustments**:
  - **Increase price** u_j if resource j is over-utilized ($s_j < 0$).

Price Coordination Interpretation (Vandenberghe)

- **System Overview**: Consider a system with B units, where each unit independently chooses its decision variable x_i, which determines how to allocate its goods.
- **Resource Constraints**: These are limits on shared resources, represented by the rows of A. Each component of the dual variable u_j represents the price of resource j.
- **Dual Update Rule**:

$$u_j^{\text{new}} = (u_j - ts_j)_+, \quad j = 1, \dots, m$$

  where $s = b - \sum_{i=1}^{B} A_i x_i$ represents the slacks.
- **Price Adjustments**:
  - **Increase price** u_j if resource j is over-utilized ($s_j < 0$).
  - **Decrease price** u_j if resource j is under-utilized ($s_j > 0$).

Price Coordination Interpretation (Vandenberghe)

- **System Overview**: Consider a system with B units, where each unit independently chooses its decision variable x_i, which determines how to allocate its goods.
- **Resource Constraints**: These are limits on shared resources, represented by the rows of A. Each component of the dual variable u_j represents the price of resource j.
- **Dual Update Rule**:

$$
u_j^{\text{new}} = (u_j - ts_j)_+, \quad j = 1, \ldots, m
$$

  where $s = b - \sum_{i=1}^{B} A_i x_i$ represents the slacks.
- **Price Adjustments**:
  - **Increase price** u_j if resource j is over-utilized ($s_j < 0$).
  - **Decrease price** u_j if resource j is under-utilized ($s_j > 0$).
  - **Never let prices get negative**; hence the use of the positive part notation $(\cdot)_+$.

Augmented Lagrangian method aka method of multipliers

Dual ascent disadvantage: convergence requires strong conditions. Augmented Lagrangian method transforms the primal problem:

$$
\begin{aligned}
&\min_x f(x) + \frac{\rho}{2}\|Ax - b\|^2 \\
&\text{s.t. } Ax = b
\end{aligned}
$$

Augmented Lagrangian method aka method of multipliers

Dual ascent disadvantage: convergence requires strong conditions. Augmented Lagrangian method transforms the primal problem:

$$
\begin{aligned}
&\min_x f(x) + \frac{\rho}{2}\|Ax - b\|^2 \\
&\text{s.t. } Ax = b
\end{aligned}
$$

where $\rho > 0$ is a parameter. This formulation is clearly equivalent to the original problem. The problem is strongly convex if matrix A has full column rank.

Augmented Lagrangian method aka method of multipliers

Dual ascent disadvantage: convergence requires strong conditions. Augmented Lagrangian method transforms the primal problem:

$$
\begin{aligned}
&\min_x f(x) + \frac{\rho}{2}\|Ax - b\|^2 \\
&\text{s.t. } Ax = b
\end{aligned}
$$

where $\rho > 0$ is a parameter. This formulation is clearly equivalent to the original problem. The problem is strongly convex if matrix A has full column rank.

Dual gradient ascent: The iterative updates are given by:

$$
\begin{aligned}
x_k &= \arg\min_x \left[f(x) + (u_{k-1})^T Ax + \frac{\rho}{2}\|Ax - b\|^2 \right] \\
u_k &= u_{k-1} + \rho(Ax_k - b)
\end{aligned}
$$

Augmented Lagrangian method aka method of multipliers

Notice step size choice $\alpha_k = \rho$ in dual algorithm. Why?

Since x_k minimizes the function:

$$f(x) + (u_{k-1})^T Ax + \frac{\rho}{2}\|Ax - b\|^2$$

over x, we have the stationarity condition:

$$0 \in \partial f(x_k) + A^T \left(u_{k-1} + \rho(Ax_k - b)\right)$$

which simplifies to:

$$\partial f(x_k) + A^T u_k$$

Augmented Lagrangian method aka method of multipliers

Notice step size choice $\alpha_k = \rho$ in dual algorithm. Why?

Since x_k minimizes the function:

$$f(x) + (u_{k-1})^T Ax + \frac{\rho}{2}\|Ax - b\|^2$$

over x, we have the stationarity condition:

$$0 \in \partial f(x_k) + A^T \left(u_{k-1} + \rho(Ax_k - b)\right)$$

which simplifies to:

$$\partial f(x_k) + A^T u_k$$

This represents the stationarity condition for the original primal problem; under mild conditions, $Ax_k - b \to 0$ as $k \to \infty$, so the KKT conditions are satisfied in the limit and x_k, u_k converge to the solutions.

- **Advantage:** The augmented Lagrangian gives better convergence.

Augmented Lagrangian method aka method of multipliers

Notice step size choice $\alpha_k = \rho$ in dual algorithm. Why?

Since x_k minimizes the function:

$$f(x) + (u_{k-1})^T Ax + \frac{\rho}{2}\|Ax - b\|^2$$

over x, we have the stationarity condition:

$$0 \in \partial f(x_k) + A^T \left(u_{k-1} + \rho(Ax_k - b)\right)$$

which simplifies to:

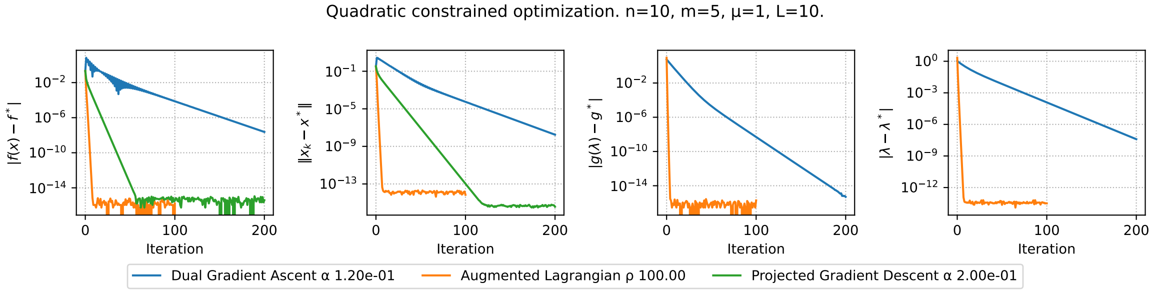
$$\partial f(x_k) + A^T u_k$$

This represents the stationarity condition for the original primal problem; under mild conditions, $Ax_k - b \to 0$ as $k \to \infty$, so the KKT conditions are satisfied in the limit and x_k, u_k converge to the solutions.

- **Advantage:** The augmented Lagrangian gives better convergence.
- **Disadvantage:** We lose decomposability! (Separability is ruined)

Example: equality constrained quadratic minimization.

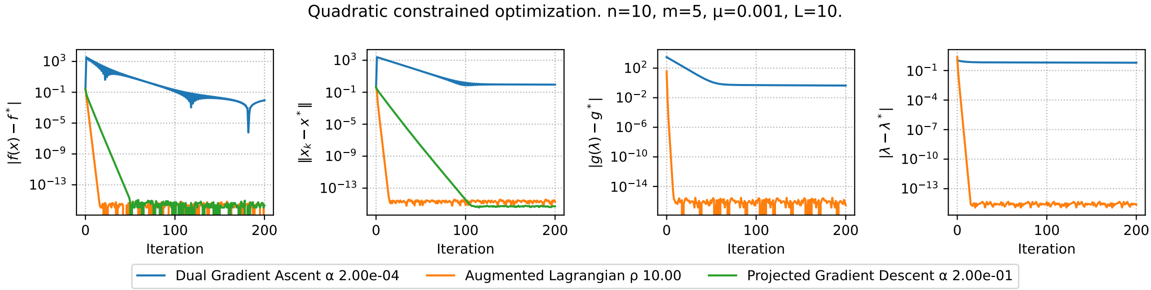
$$
f(x) = \frac{1}{2}x^T A x - b^T x \to \min_{x \in \mathbb{R}^n} \qquad \text{subject to} \quad Cx = d, \qquad A \in \mathbb{S}^n_+, C \in \mathbb{R}^{m \times n}, m < n.
$$



One can see, clear numerical superiority of the Augmented Lagrangian method both in convex and strongly convex case.

Example: equality constrained quadratic minimization.

$$f(x) = \frac{1}{2}x^T A x - b^T x \to \min_{x \in \mathbb{R}^n} \qquad \text{subject to} \quad Cx = d, \qquad A \in \mathbb{S}^n_+, C \in \mathbb{R}^{m \times n}, m < n.$$



One can see, clear numerical superiority of the Augmented Lagrangian method both in convex and strongly convex case.

Alternating Direction Method of Multipliers (ADMM)

Alternating direction method of multipliers or ADMM aims for the best of both worlds. Consider the following optimization problem:

Minimize the function:

$$\min_{x,z} f(x) + g(z)$$

$$\text{s.t. } Ax + Bz = c$$

Alternating Direction Method of Multipliers (ADMM)

Alternating direction method of multipliers or ADMM aims for the best of both worlds. Consider the following optimization problem:

Minimize the function:

$$
\begin{aligned}
&\min_{x,z} f(x) + g(z) \\
&\text{s.t. } Ax + Bz = c
\end{aligned}
$$

We augment the objective to include a penalty term for constraint violation:

$$
\begin{aligned}
&\min_{x,z} f(x) + g(z) + \frac{\rho}{2}\|Ax + Bz - c\|^2 \\
&\text{s.t. } Ax + Bz = c
\end{aligned}
$$

Alternating Direction Method of Multipliers (ADMM)

Alternating direction method of multipliers or ADMM aims for the best of both worlds. Consider the following optimization problem:

Minimize the function:

$$\min_{x,z} f(x) + g(z)$$
$$\text{s.t. } Ax + Bz = c$$

We augment the objective to include a penalty term for constraint violation:

$$\min_{x,z} f(x) + g(z) + \frac{\rho}{2}\|Ax + Bz - c\|^2$$
$$\text{s.t. } Ax + Bz = c$$

where $\rho > 0$ is a parameter. The augmented Lagrangian for this problem is defined as:

$$L_\rho(x, z, u) = f(x) + g(z) + u^T(Ax + Bz - c) + \frac{\rho}{2}\|Ax + Bz - c\|^2$$

Alternating Direction Method of Multipliers (ADMM)

ADMM repeats the following steps, for $k = 1, 2, 3, \dots$:

1. Update x:

$$x_k = \arg\min_x L_\rho(x, z_{k-1}, u_{k-1})$$

Alternating Direction Method of Multipliers (ADMM)

ADMM repeats the following steps, for $k = 1, 2, 3, \ldots$:

1. Update x:

$$x_k = \arg\min_x L_\rho(x, z_{k-1}, u_{k-1})$$

2. Update z:

$$z_k = \arg\min_z L_\rho(x_k, z, u_{k-1})$$

Alternating Direction Method of Multipliers (ADMM)

ADMM repeats the following steps, for $k = 1, 2, 3, \dots$:

1. Update x:

$$x_k = \arg\min_x L_\rho(x, z_{k-1}, u_{k-1})$$

2. Update z:

$$z_k = \arg\min_z L_\rho(x_k, z, u_{k-1})$$

3. Update u:

$$u_k = u_{k-1} + \rho(Ax_k + Bz_k - c)$$

Alternating Direction Method of Multipliers (ADMM)

ADMM repeats the following steps, for $k = 1, 2, 3, \ldots$:

1. Update x:
$$x_k = \arg\min_x L_\rho(x, z_{k-1}, u_{k-1})$$
2. Update z:
$$z_k = \arg\min_z L_\rho(x_k, z, u_{k-1})$$
3. Update u:
$$u_k = u_{k-1} + \rho(Ax_k + Bz_k - c)$$

Alternating Direction Method of Multipliers (ADMM)

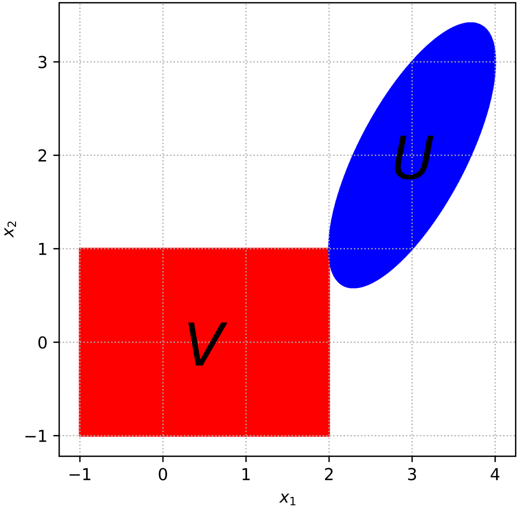
ADMM repeats the following steps, for $k = 1, 2, 3, \dots$:

1. Update x:
$$x_k = \arg\min_x L_\rho(x, z_{k-1}, u_{k-1})$$
2. Update z:
$$z_k = \arg\min_z L_\rho(x_k, z, u_{k-1})$$
3. Update u:
$$u_k = u_{k-1} + \rho(Ax_k + Bz_k - c)$$

Note: The usual method of multipliers would replace the first two steps by a joint minimization:

$$(x^{(k)}, z^{(k)}) = \arg\min_{x,z} L_\rho(x, z, u^{(k-1)})$$

Example: Alternating Projections



Consider finding a point in the intersection of convex sets $U, V \subseteq \mathbb{R}^n$:

$$\min_x I_U(x) + I_V(x)$$

To transform this problem into ADMM form, we express it as:

$$\min_{x,z} I_U(x) + I_V(z) \quad \text{subject to} \quad x - z = 0$$

Each ADMM cycle involves two projections:

$$x_k = \arg\min_x P_U\left(z_{k-1} - w_{k-1}\right)$$

$$z_k = \arg\min_z P_V\left(x_k + w_{k-1}\right)$$

$$w_k = w_{k-1} + x_k - z_k$$

Sources

- Ryan Tibshirani. Convex Optimization 10-725