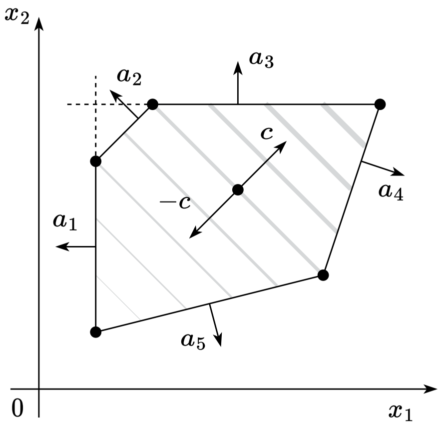
# Linear Programming. Simplex Algorithm. Applications.

Daniil Merkulov

Optimization for ML. Faculty of Computer Science. HSE University

# What is Linear Programming?




Generally speaking, all problems with linear objective and linear equalities/inequalities constraints could be considered as Linear Programming. However, there are some formulations.

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\begin{aligned}
&\min_{x \in \mathbb{R}^n} c^\top x \\
&\text{s.t. } Ax \leq b
\end{aligned}
\tag{LP.Basic}
$$

for some vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and matrix $A \in \mathbb{R}^{m \times n}$. Where the inequalities are interpreted component-wise.

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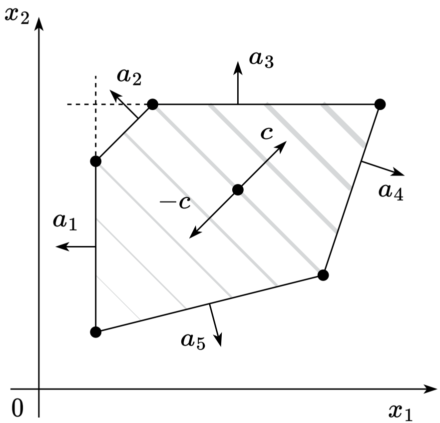


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**Standard form.** This form seems to be the most intuitive and geometric in terms of visualization. Let us have vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and matrix $A \in \mathbb{R}^{m \times n}$.

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\begin{aligned}
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& x_i \geq 0, \; i = 1, \dots, n
\end{aligned}
\tag{LP.Standard}
$$

# Example: Diet problem

$$
\begin{array}{cc}
 & 🍌\ 🍰\ 🍗\ 🥚\ 🐟 \\
\begin{array}{c} \text{Proteins} \\ \text{Carbs} \\ \text{Fats} \\ \text{Calories} \\ \text{Vitamin D} \end{array} & \left[ \begin{array}{c} \text{Amount per 100g} \\ W \in \mathbb{R}^{n \times p} \end{array} \right]
\end{array}
$$

$c \in \mathbb{R}^p$, price per 100g

$r \in \mathbb{R}^n$, nutrient requirements

$x \in \mathbb{R}^p$, amount of products, 100g

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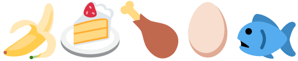

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Imagine, that you have to construct a diet plan from some set of products: bananas, cakes, chicken, eggs, fish. Each of the products has its vector of nutrients. Thus, all the food information could be processed through the matrix $W$. Let us also assume, that we have the vector of requirements for each of nutrients $r \in \mathbb{R}^n$. We need to find the cheapest configuration of the diet, which meets all the requirements:

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\text{s.t. } & Wx \succeq r \\
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Open In Colab

# Basic transformations

- Max-min

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- Unsigned variables to nonnegative variables.

$$x \leftrightarrow \begin{cases} x = x_+ - x_- \\ x_+ \geq 0 \\ x_- \geq 0 \end{cases}$$

# Example: Chebyshev approximation problem

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\min_{x \in \mathbb{R}^n} \|Ax - b\|_\infty \leftrightarrow \min_{x \in \mathbb{R}^n} \max_i |a_i^T x - b_i|
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\begin{aligned}
&\min_{t \in \mathbb{R}, x \in \mathbb{R}^n} t \\
\text{s.t. } & a_i^T x - b_i \leq t, \; i = 1, \dots, n \\
& - a_i^T x + b_i \leq t, \; i = 1, \dots, n
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$$

# $\ell_1$ approximation problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_1 \leftrightarrow \min_{x \in \mathbb{R}^n} \sum_{i=1}^n |a_i^T x - b_i|$$

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# Duality

Primal problem:

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KKT for optimal $x^*, \nu^*, \lambda^*$:

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\begin{aligned}
&L(x, \nu, \lambda) = c^T x + \nu^T (Ax - b) - \lambda^T x \\
&-A^T \nu^* + \lambda^* = c \\
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Has the following dual:

$$
\begin{aligned}
\max_{\nu \in \mathbb{R}^m} \ & -b^\top \nu \\
\text{s.t.} \ & -A^T \nu \preceq c
\end{aligned} \tag{2}
$$

Find the dual problem to the problem above (it should be the original LP). Also, write down KKT for the dual problem, to ensure, they are identical to the primal KKT.

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**PROOF.** For (i), suppose that Equation 1 has a finite optimal solution $x^*$. It follows from KKT that there are optimal vectors $\lambda^*$ and $\nu^*$ such that $(x^*, \nu^*, \lambda^*)$ satisfies KKT. We noted above that KKT for Equation 1 and Equation 2 are equivalent. Moreover, $c^T x^* = (-A^T \nu^* + \lambda^*)^T x^* = -(\nu^*)^T A x^* = -b^T \nu^*$, as claimed.

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To prove (ii), suppose that the primal is unbounded, that is, there is a sequence of points $x_k$, $k = 1, 2, 3, \ldots$ such that

$$c^T x_k \downarrow -\infty, \quad A x_k = b, \quad x_k \geq 0.$$

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$$c^T x_k \downarrow -\infty, \quad Ax_k = b, \quad x_k \geq 0.$$

Suppose too that the dual Equation 2 is feasible, that is, there exists a vector $\bar{\nu}$ such that $-A^T\bar{\nu} \leq c$. From the latter inequality together with $x_k \geq 0$, we have that $-\bar{\nu}^T A x_k \leq c^T x_k$, and therefore

$$-\bar{\nu}^T b = -\bar{\nu}^T A x_k \leq c^T x_k \downarrow -\infty,$$

yielding a contradiction. Hence, the dual must be infeasible. A similar argument can be used to show that the unboundedness of the dual implies the infeasibility of the primal.

# Example: Transportation problem

The prototypical transportation problem deals with the distribution of a commodity from a set of sources to a set of destinations. The object is to minimize total transportation costs while satisfying constraints on the supplies available at each of the sources, and satisfying demand requirements at each of the destinations.


Figure 1: Western Europe Map. Open In Colab

# Example: Transportation problem

| Customer / Source | Arnhem [€/ton] | Gouda [€/ton] | Demand [tons] |
|---|---|---|---|
| London | n/a | 2.5 | 125 |
| Berlin | 2.5 | n/a | 175 |
| Maastricht | 1.6 | 2.0 | 225 |
| Amsterdam | 1.4 | 1.0 | 250 |
| Utrecht | 0.8 | 1.0 | 225 |
| The Hague | 1.4 | 0.8 | 200 |
| **Supply [tons]** | 550 tons | 700 tons | |

$$\text{minimize:} \quad \text{Cost} = \sum_{c \in \text{Customers}} \sum_{s \in \text{Sources}} T[c,s]x[c,s]$$

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$$\sum_{s \in \text{Sources}} x[c,s] = \text{Demand}[c] \qquad \forall c \in \text{Customers}$$

This can be represented in the following graph:

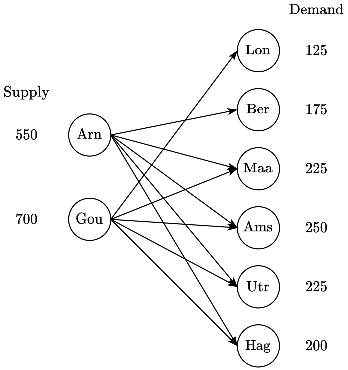
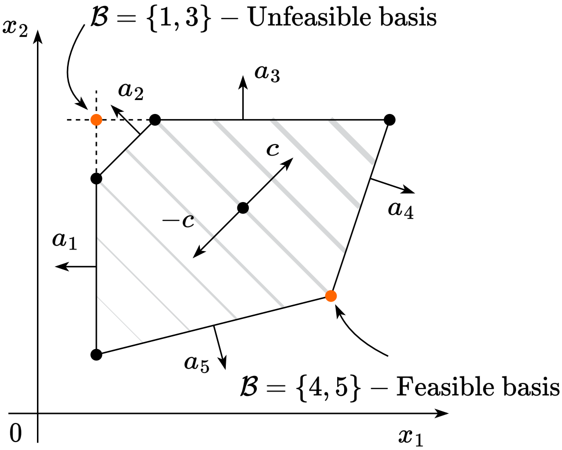



Figure 2: Graph associated with the problem

# Geometry of simplex algorithm

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- Definition: a **basis** $\mathcal{B}$ is a subset of $n$ (integer) numbers between $1$ and $m$, so that $\text{rank} A_{\mathcal{B}} = n$.

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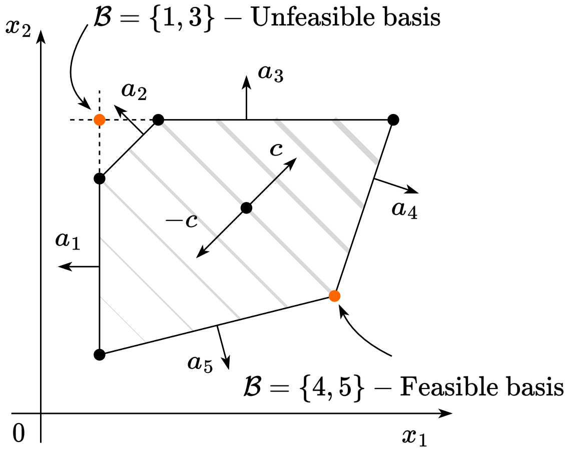


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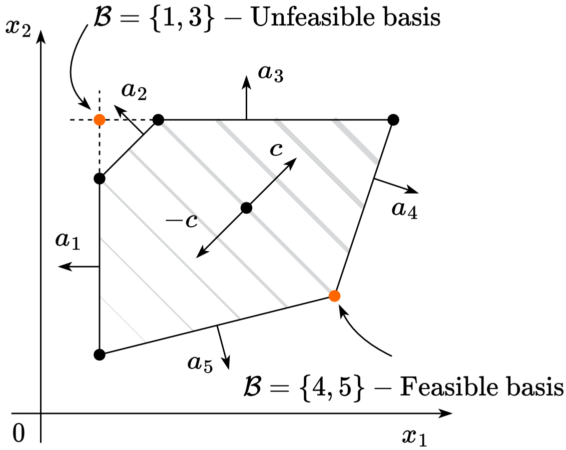


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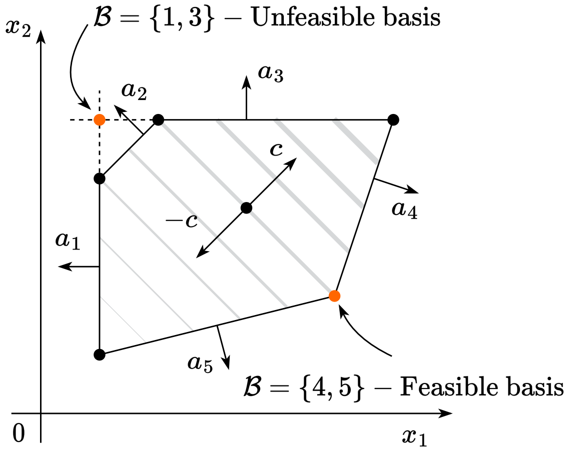


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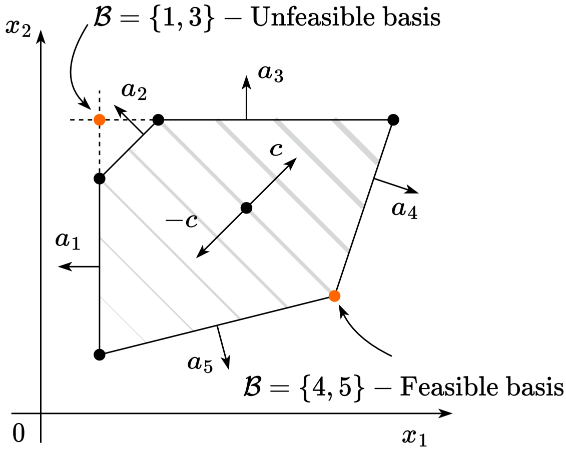


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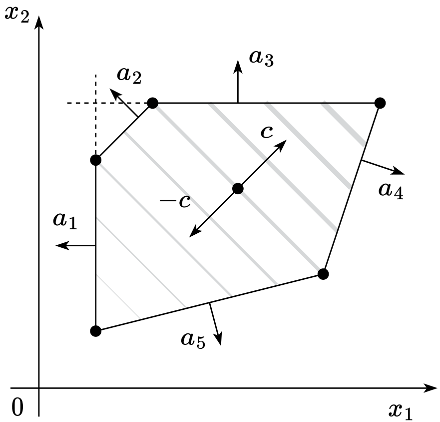
- If $Ax_{\mathcal{B}} \leq b$, then basis $\mathcal{B}$ is **feasible**.
- A basis $\mathcal{B}$ is optimal if $x_{\mathcal{B}}$ is an optimum of the LP.Inequality.

# The solution of LP if exists lies in the corner




**Theorem**

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The high-level idea of the simplex method is following:

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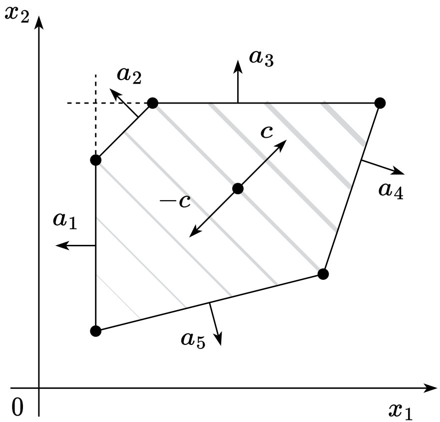


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1. If Standard LP has a nonempty feasible region, then there is at least one basic feasible point
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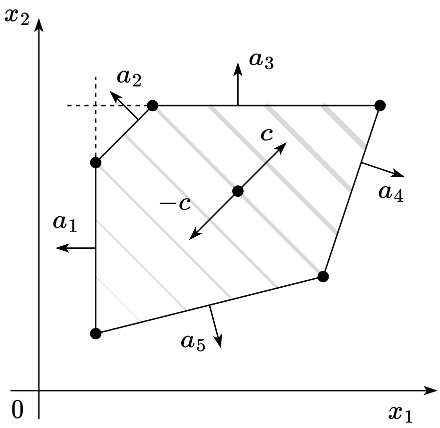


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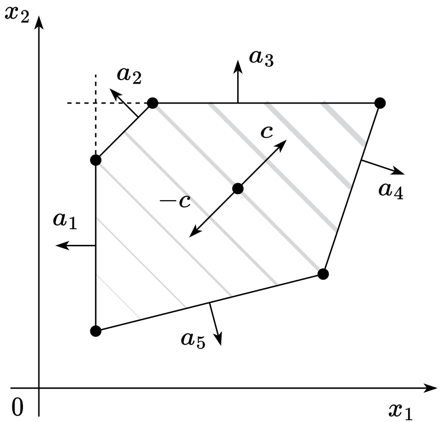


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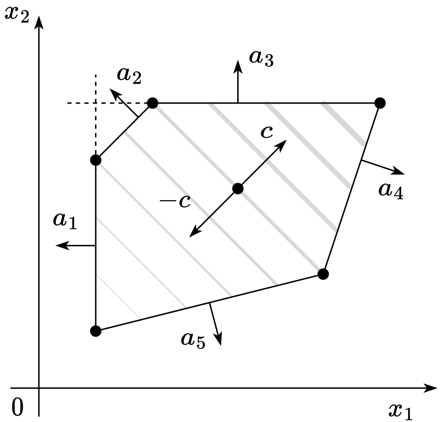


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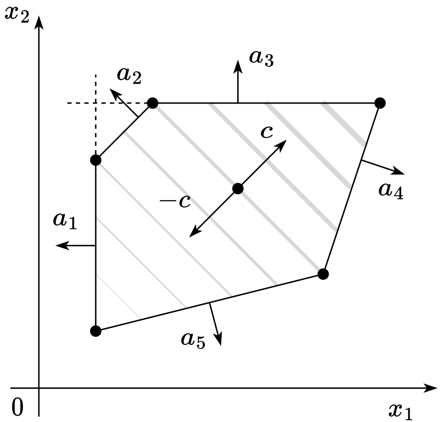
For proof see Numerical Optimization by Jorge Nocedal and Stephen J. Wright theorem 13.2

The high-level idea of the simplex method is following:

- Ensure, that you are in the corner.

# The solution of LP if exists lies in the corner




**Theorem**

1. If Standard LP has a nonempty feasible region, then there is at least one basic feasible point
2. If Standard LP has solutions, then at least one such solution is a basic optimal point.
3. If Standard LP is feasible and bounded, then it has an optimal solution.

For proof see Numerical Optimization by Jorge Nocedal and Stephen J. Wright theorem 13.2

The high-level idea of the simplex method is following:

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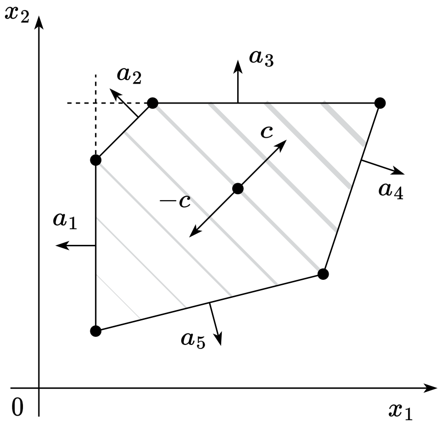


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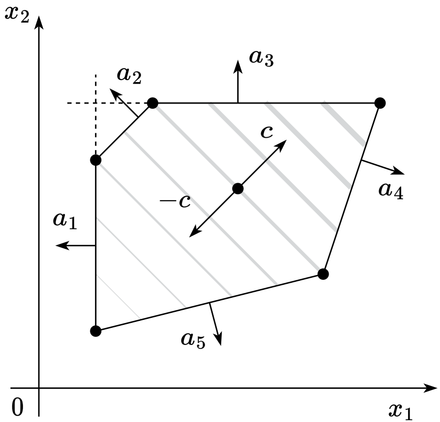


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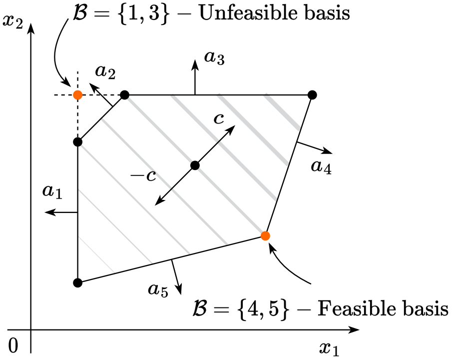
- Check optimality.
- If necessary, switch the corner (change the basis).
- Repeat until converge.

# Optimal basis




Since we have a basis, we can decompose our objective vector $c$ in this basis and find the scalar coefficients $\lambda_{\mathcal{B}}$:

$$\lambda_{\mathcal{B}}^T A_{\mathcal{B}} = c^T \leftrightarrow \lambda_{\mathcal{B}}^T = c^T A_{\mathcal{B}}^{-1}$$

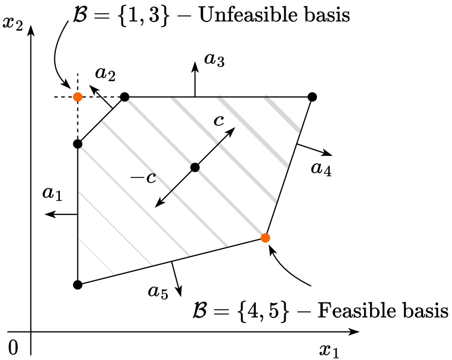
**Theorem**

If all components of $\lambda_{\mathcal{B}}$ are non-positive and $\mathcal{B}$ is feasible, then $\mathcal{B}$ is optimal.

**Proof**

$$\exists x^* : Ax^* \leq b, c^T x^* < c^T x_{\mathcal{B}}$$

# Optimal basis

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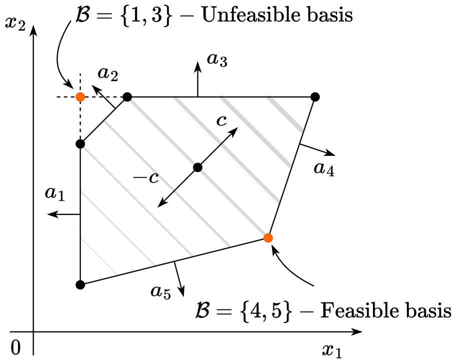


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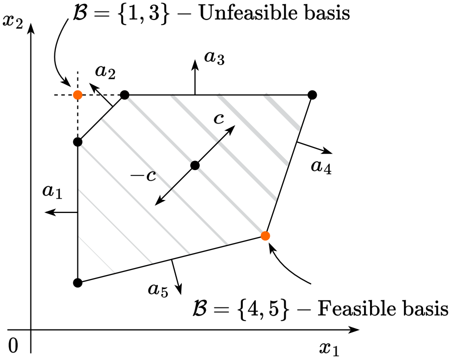


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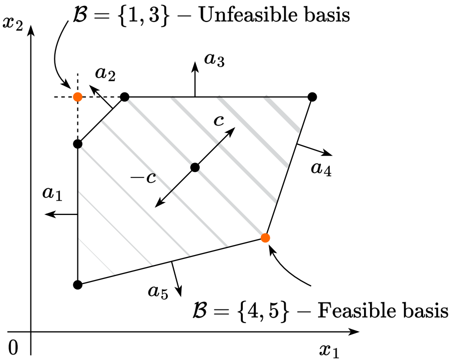


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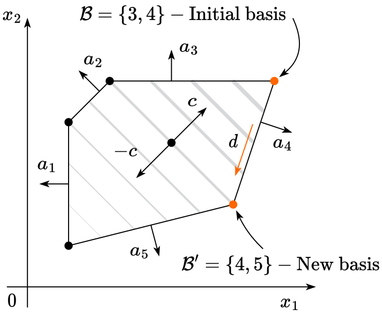
# Changing basis




Suppose, some of the coefficients of $\lambda_{\mathcal{B}}$ are positive. Then we need to go through the edge of the polytope to the new vertex (i.e., switch the basis)

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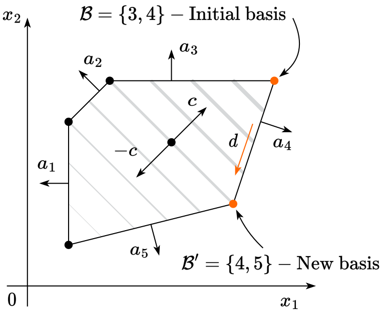


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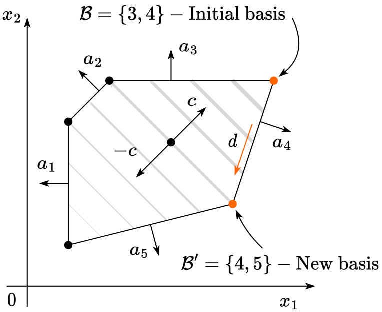


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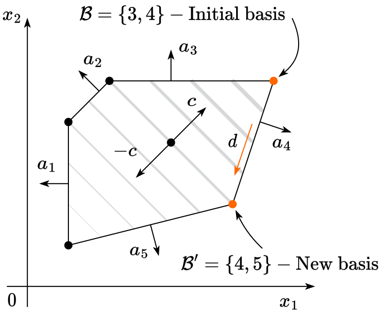


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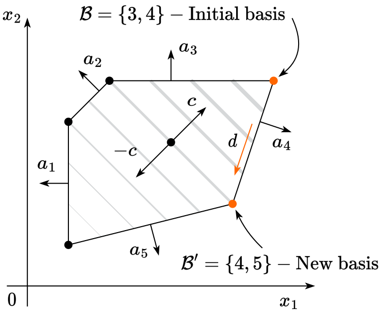


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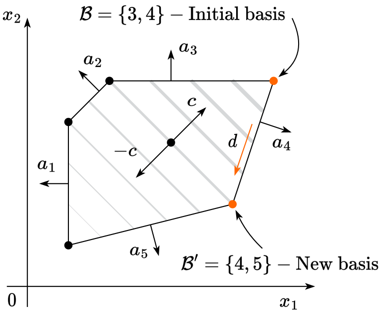


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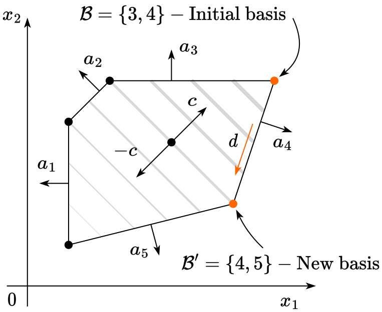


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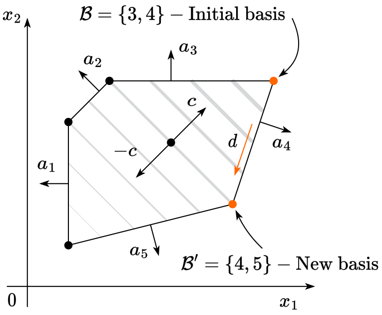


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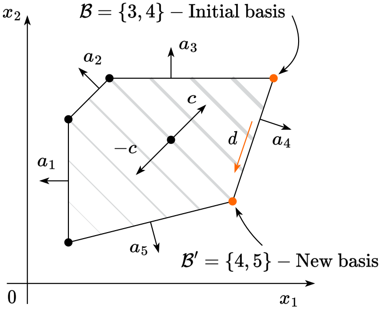


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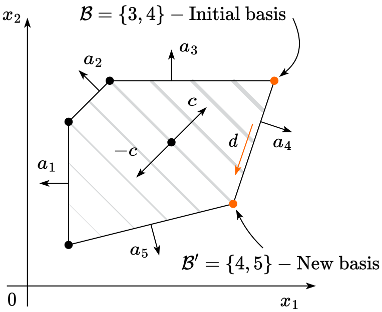
$$x_{\mathcal{B}'} = x_{\mathcal{B}} + \mu_t d = A^{-1}_{\mathcal{B}'} b_{\mathcal{B}'}$$

# Changing basis

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- Note, that changing basis implies objective function decreasing

$$c^T x_{\mathcal{B}'} = c^T (x_{\mathcal{B}} + \mu_t d) = c^T x_{\mathcal{B}} + \mu_t c^T d$$

# Finding an initial basic feasible solution

We aim to solve the following problem:

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} c^\top x \\
\text{s.t. } & Ax \leq b
\end{aligned}
\tag{3}
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The proposed algorithm requires an initial basic feasible solution and corresponding basis.

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Given the solution of Problem 4 the solution of Problem 3 can be recovered and vice versa

$$
x = y - z \qquad \Leftrightarrow \qquad y_i = \max(x_i, 0), \quad z_i = \max(-x_i, 0)
$$

Now we will try to formulate new LP problem, which solution will be basic feasible point for Problem 4. Which means, that we firstly run Simplex algorithm for Phase-1 problem and run Phase-2 problem with known starting point. Note, that basic feasible solution for Phase-1 should be somehow easily established.

# Finding an initial basic feasible solution

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\begin{aligned}
&\min_{\xi \in \mathbb{R}^m, y \in \mathbb{R}^n, z \in \mathbb{R}^n} \sum_{i=1}^{m} \xi_i \\
\text{s.t. } & Ay - Az \le b + \xi \\
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- Now we know, that if we can solve a Phase-1 problem then we will either find a starting point for the simplex method in the original method (if slacks are zero) or verify that the original problem was infeasible (if slacks are non-zero).

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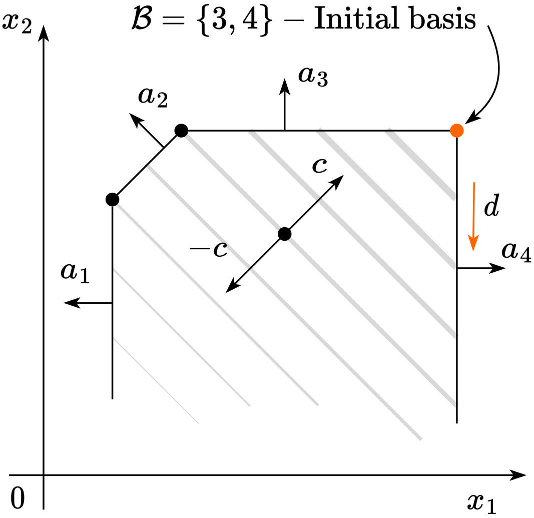
- If Phase-1 optimum is zero (i.e. all slacks $\xi_i$ are zero), then we get a feasible basis for Phase-2.
  **Proof:** trivial check.

- Now we know, that if we can solve a Phase-1 problem then we will either find a starting point for the simplex method in the original method (if slacks are zero) or verify that the original problem was infeasible (if slacks are non-zero).
- But how to solve Phase-1? It has basic feasible solution (the problem has $2n + m$ variables and the point below ensures $2n + m$ inequalities are satisfied as equalities (active).)

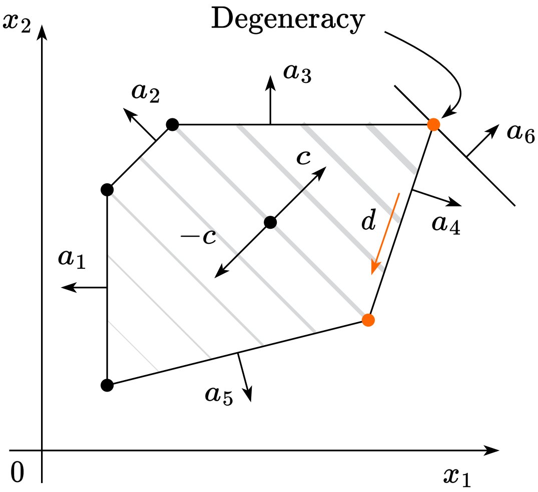
$$
z = 0 \quad y = 0 \quad \xi_i = \max(0, -b_i)
$$

# Unbounded budget set



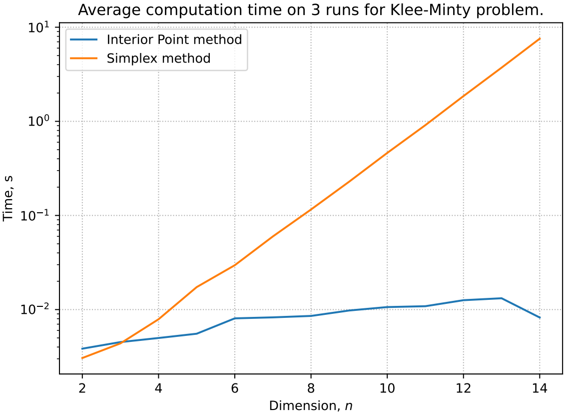

In this case, all $\mu_j$ will be negative.

# Degeneracy




One needs to handle degenerate corners carefully. If no degeneracy exists, one can guarantee a monotonic decrease of the objective function on each iteration.

# Exponential convergence




- A wide variety of applications could be formulated as linear programming.

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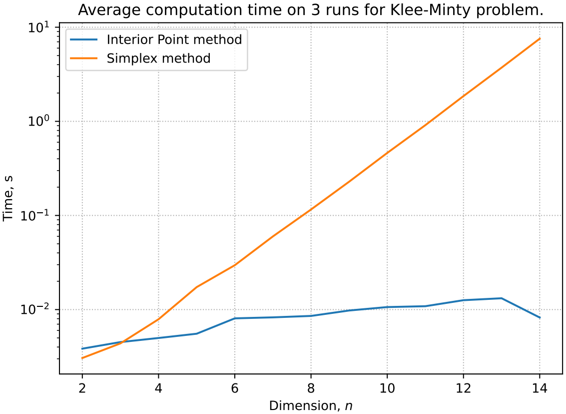


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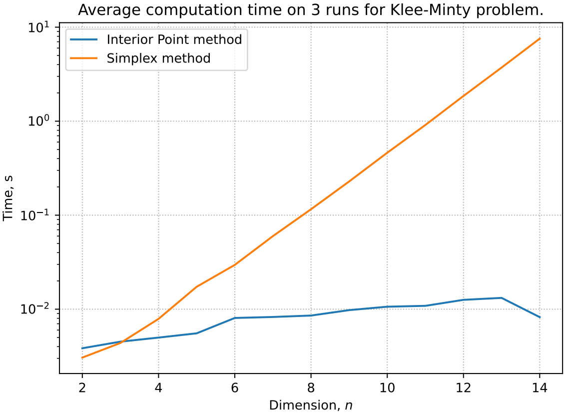


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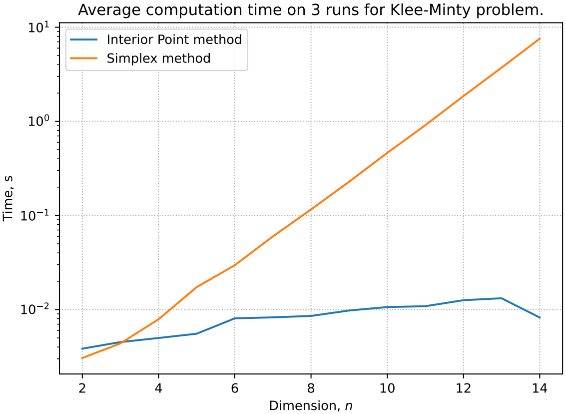


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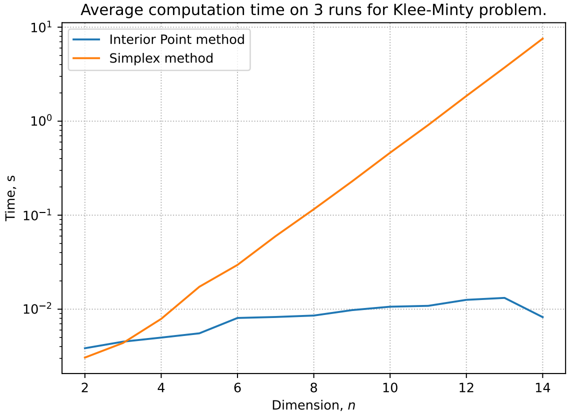


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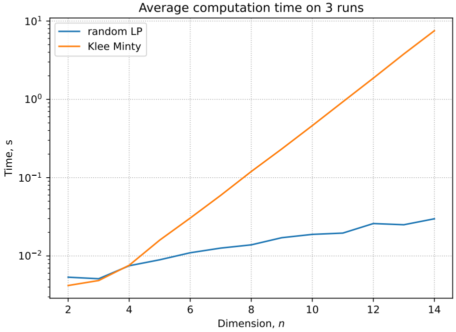
- Major breakthrough - Narendra Karmarkar's method for solving LP (1984) using interior point method.
- Interior point methods are the last word in this area. However, good implementations of simplex-based methods and interior point methods are similar for routine applications of linear programming.

# Klee Minty example

Since the number of edge points is finite, the algorithm should converge (except for some degenerate cases, which are not covered here). However, the convergence could be exponentially slow, due to the high number of edges. There is the following iconic example when the simplex algorithm should perform exactly all vertexes.

In the following problem, the simplex algorithm needs to check $2^n - 1$ vertexes with $x_0 = 0$.

$$
\begin{align*}
& \max_{x \in \mathbb{R}^n} 2^{n-1}x_1 + 2^{n-2}x_2 + \dots + 2x_{n-1} + x_n \\
\text{s.t. } & x_1 \leq 5 \\
& 4x_1 + x_2 \leq 25 \\
& 8x_1 + 4x_2 + x_3 \leq 125 \\
& \ldots \\
& 2^n x_1 + 2^{n-1}x_2 + 2^{n-2}x_3 + \ldots + x_n \leq 5^n \\
& x \geq 0
\end{align*}
$$



# Minimization of convex function as LP

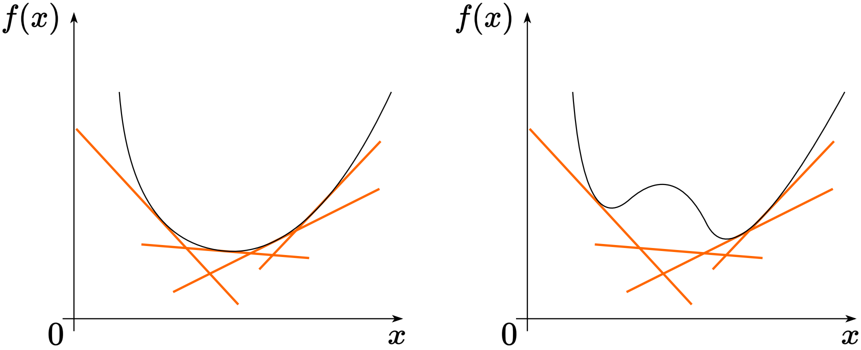



Figure 3: How LP can help with general convex problem

- The function is convex iff it can be represented as a pointwise maximum of linear functions.

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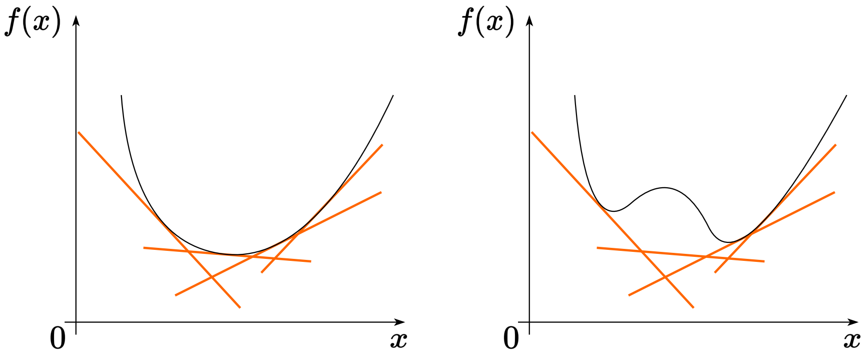


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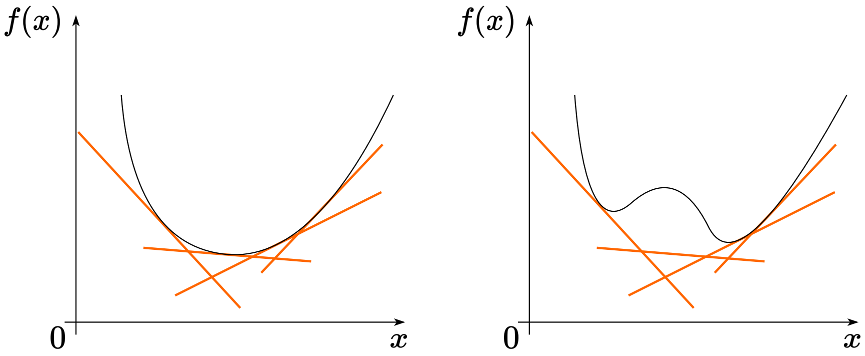


Figure 3: How LP can help with general convex problem

- The function is convex iff it can be represented as a pointwise maximum of linear functions.
- In high dimensions, the approximation may require too many functions.
- More efficient convex optimizers (not reducing to LP) exist.

# Complexity of MIP

Consider the following Mixed Integer Programming (MIP):

$$
\begin{aligned}
z = 8x_1 + 11x_2 + 6x_3 + 4x_4 \to &\max_{x_1, x_2, x_3, x_4} \\
\text{s.t. } 5x_1 + 7x_2 + 4x_3 + 3x_4 &\leq 14 \\
x_i \in \{0, 1\} \quad &\forall i
\end{aligned}
\tag{5}
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Consider the following Mixed Integer Programming (MIP): Relax it to:

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\begin{split}
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Optimal solution

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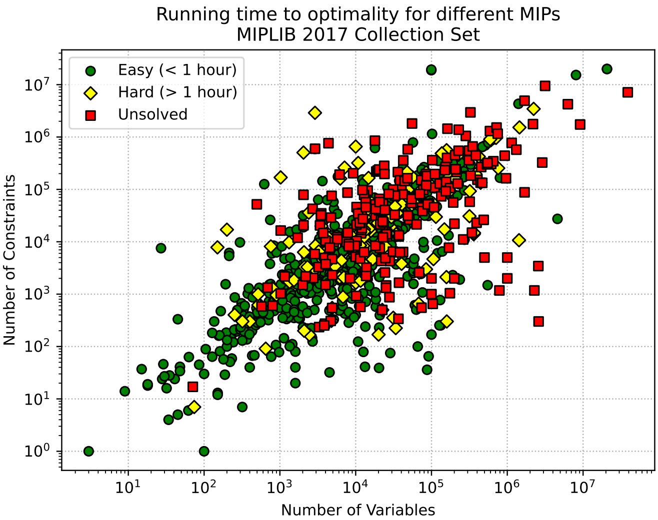
- Naive rounding of LP relaxation of the initial MIP problem might lead to infeasible or suboptimal solution.
- General MIP is NP-hard.
- However, if the coefficient matrix of an MIP is a *totally unimodular matrix*, then it can be solved in polynomial time.

# Unpredictable complexity of MIP

- It is hard to predict what will be solved quickly and what will take a long time



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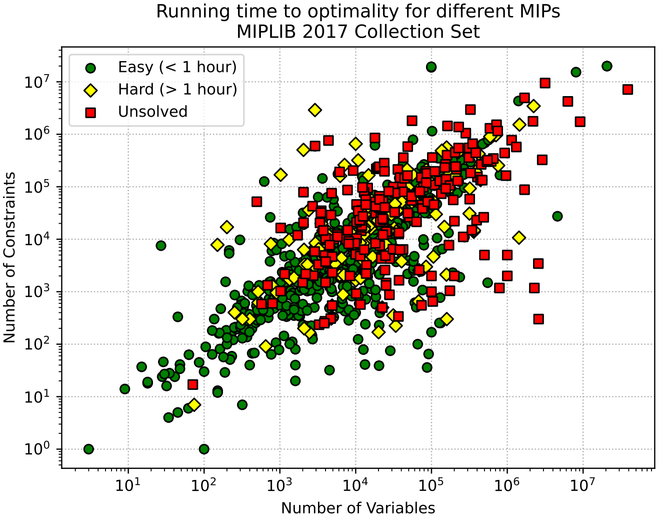

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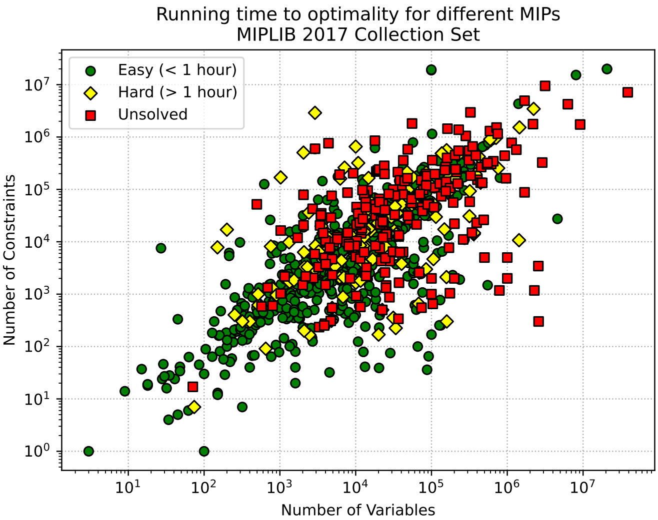

# Hardware progress vs Software progress

What would you choose, assuming, that the question posed correctly (you can compile software for any hardware and the problem is the same for both options)? We will consider the time period from 1992 to 2023.

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Solving MIP with an old software on the modern hardware

**Software**

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Moore's law states, that computational power doubles every 18 monthes.

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R. Bixby conducted an intensive experiment with benchmarking all CPLEX software version starting from 1992 to 2007 and measured overall software progress $(29000$ times), later (in 2009) he was a cofounder of Gurobi optimization software, which gives additional $\approx 81$ speedup on MILP.

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It turns out that if you need to solve a MILP, it is better to use an old computer and modern methods than vice versa, the newest computer and methods of the early 1990s![1]

[1] R. Bixby report Recent study