Duality

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The reader will find no figures in this work. The methods which I set forth do not require either constructions or geometrical or mechanical reasonings: but only algebraic operations, subject to a regular and uniform rule of procedure.

OR PROOFS

Preface to Mécanique analytique

Figure 1: Joseph-Louis Lagrange

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Motivation

Duality lets us associate to any constrained optimization problem a concave maximization problem, whose solutions lower bound the optimal value of the original problem. What is interesting is that there are cases, when one can solve the primal problem by first solving the dual one. Now, consider a general constrained optimization problem:

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Dual: $g(y) \to \max\limits_{y \in \Omega}$

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As a consequence:

$$\max_{y \in \Omega} g(y) \leq \min_{x \in S} f(x)$$

Lagrange duality

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\text{s.t. } & f_i(x) \leq 0, \; i = 1, \ldots, m \\
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$$

And the Lagrangian, associated with this problem:

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L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x) = f_0(x) + \lambda^\top f(x) + \nu^\top h(x)
$$

Dual function

We assume $\mathcal{D} = \bigcap_{i=0}^{m} \textbf{dom } f_i \cap \bigcap_{i=1}^{p} \textbf{dom } h_i$ is nonempty. We define the Lagrange dual function (or just dual function) $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ as the minimum value of the Lagrangian over x: for $\lambda \in \mathbb{R}^m, \nu \in \mathbb{R}^p$

Dual function

ИНФИМУМ

$h(x) = 0 \iff \begin{cases} h(x) \leq 0 \\ -h(x) \leq 0 \end{cases}$

$\min f_0(x)$

$f_i(x) \leq 0$

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g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)
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вогнутая

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When the Lagrangian is unbounded below in x, the dual function takes on the value $-\infty$. Since the dual function is the pointwise infimum of a family of affine functions of (λ, ν), it is concave, even when the original problem is not convex.

Dual function as a lower bound

Let us show, that the dual function yields lower bounds on the optimal value p^* of the original problem for any $\lambda \succeq 0, \nu$. Suppose some $\hat{x}$ is a feasible point for the original problem, i.e., $f_i(\hat{x}) \leq 0$ and $h_i(\hat{x}) = 0, \ \lambda \succeq 0$. Then we have:

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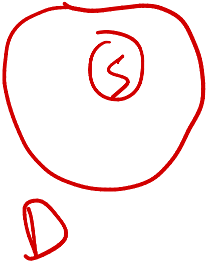

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\boxed{g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu)} \leq L(\hat{x}, \lambda, \nu) \leq f_0(\hat{x})
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$p^* := \min_{x \in S} f(x)$

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$$g(\lambda, \nu) \to \max_{\lambda \in \mathbb{R}^m,\ \nu \in \mathbb{R}^p}$$

s.t. $\lambda \succeq 0$

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двойственная задача

$$g(\lambda, \nu) \to \max_{\lambda \in \mathbb{R}^m, \; \nu \in \mathbb{R}^p}$$

s.t. $\lambda \succeq 0$

The term "dual feasible", to describe a pair (λ, ν) with $\lambda \succeq 0$ and $g(\lambda, \nu) > -\infty$, now makes sense. It means, as the name implies, that (λ, ν) is feasible for the dual problem. We refer to (λ^*, ν^*) as dual optimal or optimal Lagrange multipliers if they are optimal for the above problem.

Summary

	Primal	Dual
Function	$f_0(x)$	$g(\lambda, \nu) = \min\limits_{x \in \mathcal{D}} L(x, \lambda, \nu)$
Variables	$x \in S \subseteq \mathbb{R}^{\ltimes}$	$\lambda \in \mathbb{R}^m_+, \nu \in \mathbb{R}^p$
Constraints	$f_i(x) \leq 0,\ i = 1, \ldots, m$ $h_i(x) = 0,\ i = 1, \ldots, p$	$\lambda_i \geq 0, \forall i \in \overline{1, m}$
Problem	$f_0(x) \to \min\limits_{x \in \mathbb{R}^n}$ s.t. $f_i(x) \leq 0,\ i = 1, \ldots, m$ $h_i(x) = 0,\ i = 1, \ldots, p$	$g(\lambda, \nu) \to \max\limits_{\lambda \in \mathbb{R}^m, \nu \in \mathbb{R}^p}$ s.t. $\lambda \succeq 0$
Optimal	x^* if feasible, $p^* = f_0(x^*)$	λ^*, ν^* if $\max$ is achieved, $d^* = g(\lambda^*, \nu^*)$

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We are addressing a problem within a non-empty budget set, defined as follows:

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\begin{aligned}
\text{min} \quad & x^T x \\
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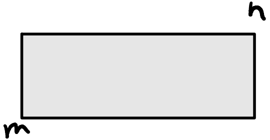
$$

$= \|x\|^2$



with the matrix $A \in \mathbb{R}^{m \times n}$.

1. $L(x, \nu) = x^T x + \nu^T (Ax - b)$

$$\frac{\partial L}{\partial x} = 2x + A^T \nu = 0$$

$2x$: $n \times 1$

$\nu^T A$: $1 \times m$, $m \times n$ → $1 \times n$

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This problem is devoid of inequality constraints, presenting m linear equality constraints instead. The Lagrangian is expressed as $L(x, \nu) = x^T x + \nu^T (Ax - b)$, spanning the domain $\mathbb{R}^n \times \mathbb{R}^m$. The dual function is denoted by $g(\nu) = \inf_x L(x, \nu)$. Given that $L(x, \nu)$ manifests as a convex quadratic function in terms of x, the minimizing x can be derived from the optimality condition

$$g(\nu) = \inf_{x \in D} L(x, \nu) = \inf_{x \in \mathbb{R}^n} x^T x + \nu^T (Ax - b)$$

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$$\hat{x} = -\frac{1}{2} A^T \nu$$

$$\Rightarrow g(\nu) = L(\hat{x}, \nu) =$$

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leading to $x = -(1/2) A^T \nu$. As a result, the dual function is articulated as

$$+\frac{1}{4} \nu^T A A^T \nu + \nu^T \left(A \left(-\frac{1}{2} A^T \nu \right) - b \right)$$

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$$
-(1/4)\nu^T A A^T \nu - b^T \nu \leq \inf\{x^T x \mid Ax = b\}.
$$

Which is a simple non-trivial lower bound without any problem solving.

Example. Two-way partitioning problem

We are examining a (nonconvex) problem:

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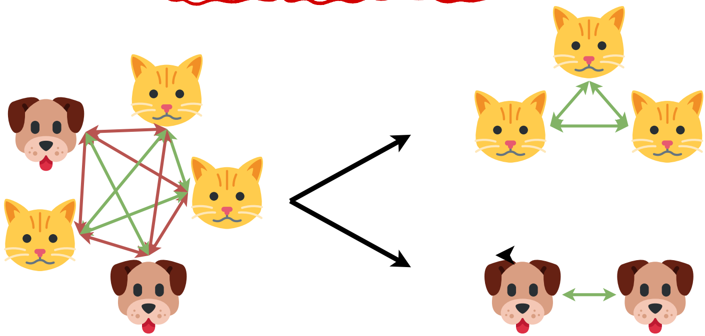

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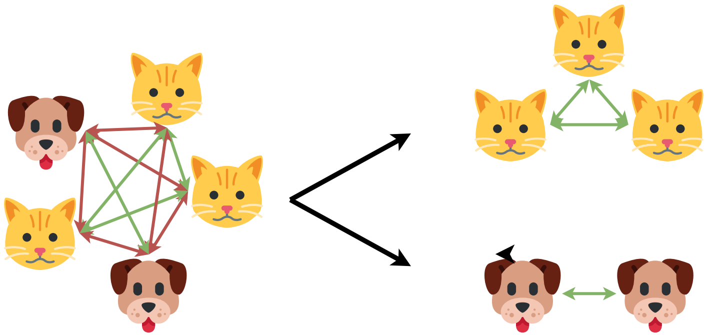

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This problem can be construed as a two-way partitioning problem over a set of n elements, denoted as $\{1, \ldots, n\}$: A viable x corresponds to the partition

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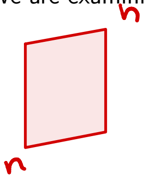

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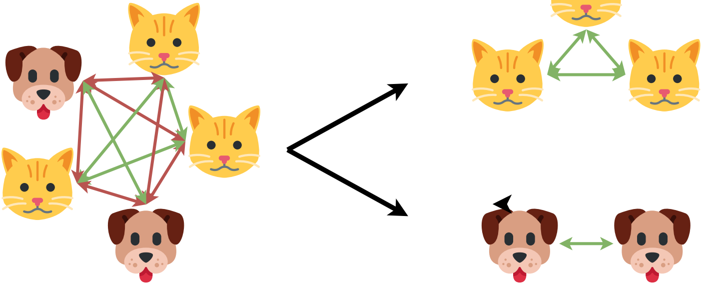
The coefficient W_{ij} in the matrix represents the expense associated with placing elements i and j in the same partition, while $-W_{ij}$ signifies the cost of segregating them. The objective encapsulates the aggregate cost across all pairs of elements, and the challenge posed by problem is to find the partition that minimizes the total cost.



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We now derive the dual function for this problem. The Lagrangian is expressed as

$$L(x, \nu) = x^T W x + \sum_{i=1}^{n} \nu_i (x_i^2 - 1) = x^T (W + \text{diag}(\nu)) x - \mathbf{1}^T \nu.$$

. . .

By minimizing over x, we procure the Lagrange dual function:

$$g(\nu) = \inf_x x^T (W + \text{diag}(\nu)) x - \mathbf{1}^T \nu = \begin{cases} -\mathbf{1}^T \nu & \text{if } W + \text{diag}(\nu) \succeq 0 \\ -\infty & \text{otherwise,} \end{cases}$$

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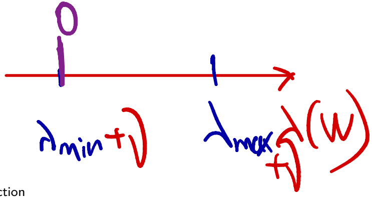
This dual function furnishes lower bounds on the optimal value of the problem. For instance, we can adopt the particular value of the dual variable

$$\nu = -\lambda_{\min}(W)\mathbf{1}$$



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exploiting the realization that the infimum of a quadratic form is either zero (when the form is positive semidefinite) or $-\infty$ (when it's not).

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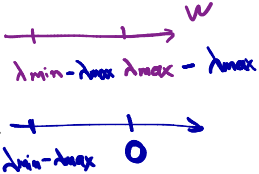

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The code for the problem is available here Open in Colab

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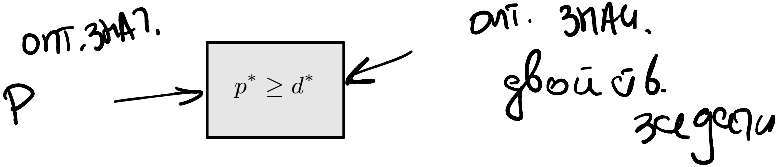

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ЗАЗОР двойственности

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Strong duality in linear least squares

$$g(\nu) = -\frac{1}{4}\nu^T A A^T \nu - b^T \nu$$

$x^T x \to \min$

$Ax = b$

$m < n$

Exercise

In the Least-squares solution of linear equations example above calculate the primal optimum p^* and the dual optimum d^* and check whether this problem has strong duality or not.

$g(\nu) \to \max_{\nu}$

$\frac{1}{4}\nu^T A A^T \nu + b^T \nu \to \min_{\nu}$

$\frac{1}{4} \cdot 2 \cdot A A^T \cdot \nu + b = 0 \to \boxed{\nu^* = -2(\underset{m\,m}{AA^T})^{-1} b}$

$$d^* = g(\nu^*) = \frac{1}{4}\left(4(AA^T)^{-1}b\right)^T AA^T (AA^T)^{-1} b - b^T \cdot 2 (AA^T)^{-1} b$$

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$$2x + A^T\nu = 0$$

$$x = -\frac{1}{2}A^T\nu = A^T(AA^T)^{-1}b$$

$$p^* = x^Tx = x^{*T}\cdot x^* = b^T(AA^T)^{-1}b = d^*$$

есть сильн. двойств.

Useful features of duality

- **Construction of lower bound on solution of the primal problem.**

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- **Dual function is always concave**

  As a pointwise minimum of affine functions.

Slater's condition

Theorem

If for a convex optimization problem (i.e., assuming minimization, f_0, f_i are convex and h_i are affine), there exists a point x such that $h(x) = 0$ and $f_i(x) < 0$ (existance of a strictly feasible point), then we have a zero duality gap and KKT conditions become necessary and sufficient.

An example of convex problem, when Slater's condition does not hold

Example

$$\min\{f_0(x) = x \mid f_1(x) = \frac{x^2}{2} \leq 0\},$$

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$$\min\{f_0(x) = x \mid f_1(x) = \frac{x^2}{2} \leq 0\},$$

The only point in the budget set is: $x^* = 0$. However, it is impossible to find a non-negative $\lambda^* \geq 0$, such that

$$\nabla f_0(0) + \lambda^* \nabla f_1(0) = \boxed{1 + \lambda^* x = 0.}$$

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Sensitivity analysis

Let us switch from the original optimization problem

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\text{s.t. } & f_i(x) \leq 0, \; i = 1, \ldots, m \\
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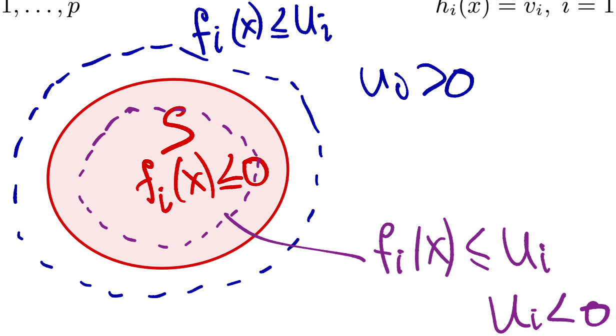

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\end{aligned}
\tag{Per}
$$

Note, that we still have the only variable $x \in \mathbb{R}^n$, while treating $u \in \mathbb{R}^m, v \in \mathbb{R}^p$ as parameters. It is obvious, that Per$(u, v) \to$ P if $u = 0, v = 0$. We will denote the optimal value of Per as $p^*(u, v)$, while the optimal value of the original problem P is just p^*. One can immediately say, that $p^*(u, v) = p^*$.

Speaking of the value of some i-th constraint we can say, that

- $u_i = 0$ leaves the original problem
- $u_i > 0$ means that we have relaxed the inequality
- $u_i < 0$ means that we have tightened the constraint

Sensitivity analysis

Let us switch from the original optimization problem

$$\begin{aligned} f_0(x) &\to \min_{x \in \mathbb{R}^n} \\ \text{s.t. } f_i(x) &\leq 0, \; i = 1, \ldots, m \\ h_i(x) &= 0, \; i = 1, \ldots, p \end{aligned} \tag{P}$$

To the perturbed version of it:

$$\begin{aligned} f_0(x) &\to \min_{x \in \mathbb{R}^n} \\ \text{s.t. } f_i(x) &\leq u_i, \; i = 1, \ldots, m \\ h_i(x) &= v_i, \; i = 1, \ldots, p \end{aligned} \tag{Per}$$

Note, that we still have the only variable $x \in \mathbb{R}^n$, while treating $u \in \mathbb{R}^m, v \in \mathbb{R}^p$ as parameters. It is obvious, that Per$(u, v) \to$ P if $u = 0, v = 0$. We will denote the optimal value of Per as $p^*(u, v)$, while the optimal value of the original problem P is just p^*. One can immediately say, that $p^*(u, v) = p^*$.

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One can even show, that when P is convex optimization problem, $p^*(u, v)$ is a convex function.

Sensitivity analysis

Suppose, that strong duality holds for the orriginal problem and suppose, that x is any feasible point for the perturbed problem:

$$
\boxed{p^*(0,0) = p^*} = d^* = g(\lambda^*, \nu^*) \leq
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Sensitivity analysis

Suppose, that strong duality holds for the orriginal problem and suppose, that x is any feasible point for the perturbed problem:

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p^*(0,0) = p^* = d^* = g(\lambda^*, \nu^*) \leq \\
\leq L(x, \lambda^*, \nu^*) =
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\end{aligned}
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Sensitivity analysis

$f_i(x) \leq u_i$

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And taking the optimal x for the perturbed problem, we have:

$$
p^*(u,v) \geq p^*(0,0) - {\lambda^*}^T u - {\nu^*}^T v \tag{1}
$$

Sensitivity analysis

In scenarios where strong duality holds, we can draw several insights about the sensitivity of optimal solutions in relation to the Lagrange multipliers. These insights are derived from the inequality expressed in equation above:

- **Impact of Tightening a Constraint (Large $\lambda_i^\star$)**:
  When the ith constraint's Lagrange multiplier, $\lambda_i^\star$, holds a substantial value, and if this constraint is tightened (choosing $u_i < 0$), there is a guarantee that the optimal value, denoted by $p^\star(u, v)$, will significantly increase.

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These interpretations provide a framework for understanding how changes in constraints, reflected through their corresponding Lagrange multipliers, impact the optimal solution in problems where strong duality holds.

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The same idea can be used to establish the fact about v_i. The local sensitivity result Equation 2 provides a way to understand the impact of constraints on the optimal solution x^* of an optimization problem. If a constraint $f_i(x^*)$ is negative at x^*, it's not affecting the optimal solution, meaning small changes to this constraint won't alter the optimal value. In this case, the corresponding optimal Lagrange multiplier will be zero, as per the principle of complementary slackness.

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However, if $f_i(x^*) = 0$, meaning the constraint is precisely met at the optimum, then the situation is different. The value of the i-th optimal Lagrange multiplier, λ_i^*, gives us insight into how 'sensitive' or 'active' this constraint is. A small λ_i^* indicates that slight adjustments to the constraint won't significantly affect the optimal value. Conversely, a large λ_i^* implies that even minor changes to the constraint can have a significant impact on the optimal solution.

Mixed strategies for matrix games

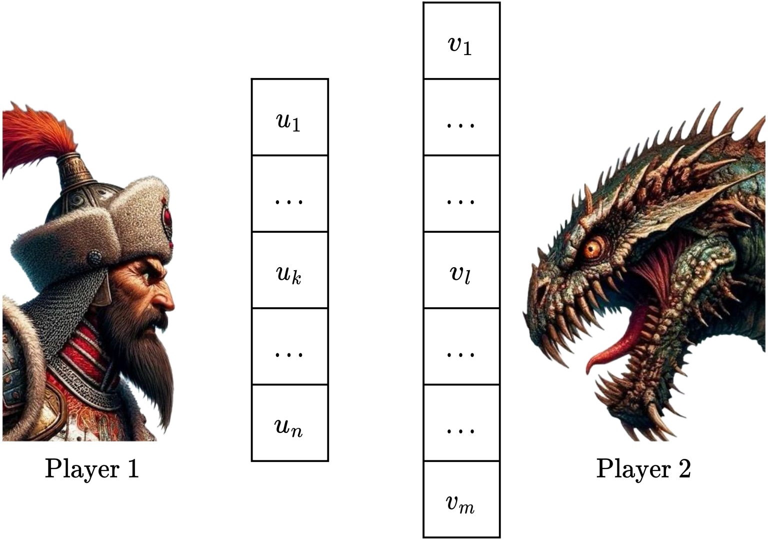

Figure 3: The scheme of a mixed strategy matrix game

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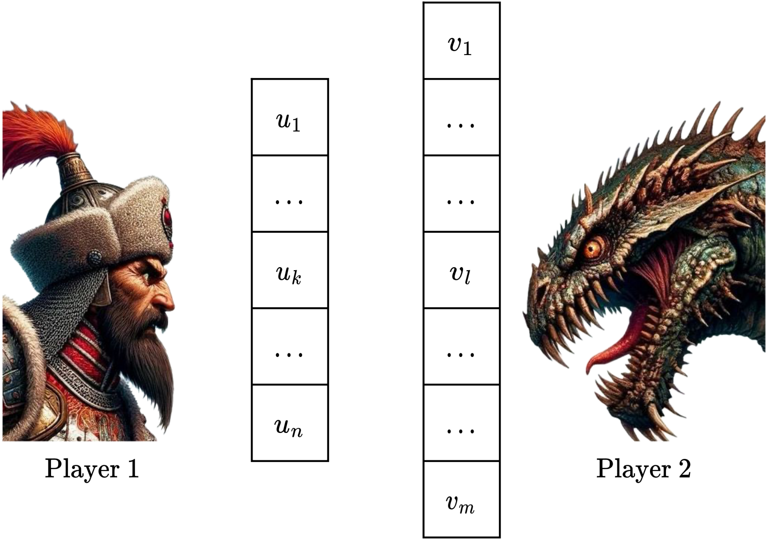

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In zero-sum matrix games, players 1 and 2 choose actions from sets $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$, respectively. The outcome is a payment from player 1 to player 2, determined by a payoff matrix $P \in \mathbb{R}^{n \times m}$. Each player aims to use mixed strategies, choosing actions according to a probability distribution: player 1 uses probabilities u_k for each action i, and player 2 uses v_l.

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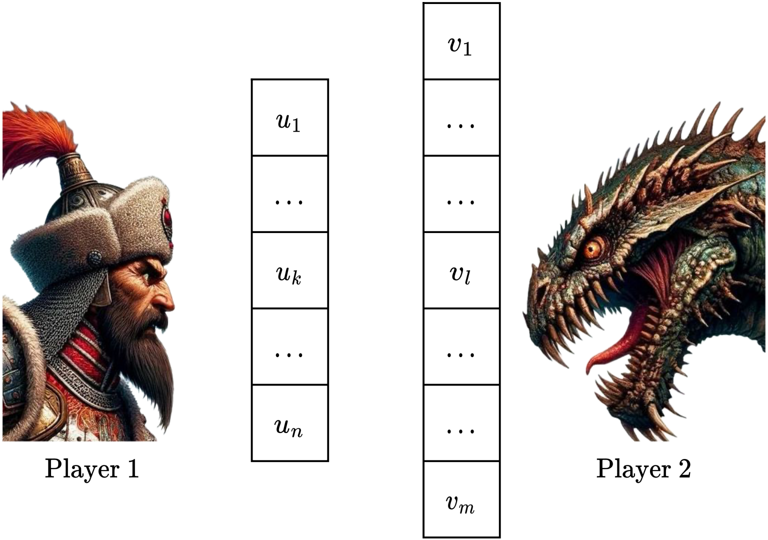

Figure 3: The scheme of a mixed strategy matrix game

In zero-sum matrix games, players 1 and 2 choose actions from sets $\{1, \dots, n\}$ and $\{1, \dots, m\}$, respectively. The outcome is a payment from player 1 to player 2, determined by a payoff matrix $P \in \mathbb{R}^{n \times m}$. Each player aims to use mixed strategies, choosing actions according to a probability distribution: player 1 uses probabilities u_k for each action i, and player 2 uses v_l.

The expected payoff from player 1 to player 2 is given by $\sum_{k=1}^{n} \sum_{l=1}^{m} u_k v_l P_{kl} = u^T P v$. Player 1 seeks to minimize this expected payoff, while player 2 aims to maximize it.

Mixed strategies for matrix games. Player 1's Perspective

Player 1

u_1

$\dots$

u_k

$\dots$

u_n

Assuming player 2 knows player 1's strategy u, player 2 will choose v to maximize $u^T P v$. The worst-case expected payoff is thus:

$$\max_{v \geq 0, 1^T v = 1} u^T P v = \max_{i=1,\dots,m} (P^T u)_i$$

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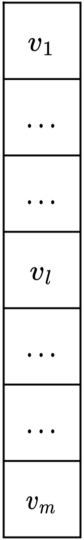
Player 1's optimal strategy minimizes this worst-case payoff, leading to the optimization problem:

$$\begin{aligned} &\min \max_{i=1,\dots,m} (P^T u)_i \\ &\text{s.t. } u \geq 0 \\ &1^T u = 1 \end{aligned} \tag{3}$$

This forms a convex optimization problem with the optimal value denoted as p_1^*.

Mixed strategies for matrix games. Player 2's Perspective



Player 2

Conversely, if player 1 knows player 2's strategy v, the goal is to minimize $u^T P v$. This leads to:

$$\min_{u \geq 0, 1^T u = 1} u^T P v = \min_{i=1,\dots,n} (Pv)_i$$

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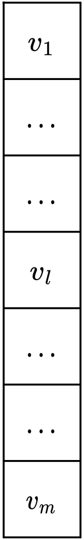

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$$\min_{u \geq 0, 1^T u = 1} u^T P v = \min_{i=1,\dots,n} (Pv)_i$$

Player 2 then maximizes this to get the largest guaranteed payoff, solving the optimization problem:

$$\begin{aligned} &\max \min_{i=1,\dots,n} (Pv)_i \\ &\text{s.t. } v \geq 0 \\ &1^T v = 1 \end{aligned} \tag{4}$$

The optimal value here is p_2^*.

Mixed strategies for matrix games

Duality and Equivalence

It's generally advantageous to know the opponent's strategy, but surprisingly, in mixed strategy matrix games, this advantage disappears. The key lies in duality: the problems above are Lagrange duals. By formulating player 1's problem as a linear program and introducing Lagrange multipliers, we find that the dual problem matches player 2's problem. Due to strong duality in feasible linear programs, $p_1^* = p_2^*$, showing no advantage in knowing the opponent's strategy.

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Conclusion

This formulation shows that the Lagrange dual problem is equivalent to problem Equation 4. Given the feasibility of these linear programs, strong duality holds, meaning the optimal values of Equation 3 and Equation 4 are equal.