Strongly convex functions. Optimality conditions.

Seminar

Optimization for ML. Faculty of Computer Science. HSE University

Convex Function

The function $f(x)$, **which is defined on the convex set** $S \subseteq \mathbb{R}^n$, is called **convex** on S, if:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$$

for any $x_1, x_2 \in S$ and $0 \leq \lambda \leq 1$.
If the above inequality holds as strict inequality $x_1 \neq x_2$ and $0 < \lambda < 1$, then the function is called **strictly convex** on S.

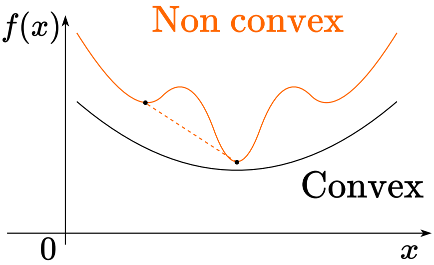


Figure 1: Difference between convex and non-convex function

Strong Convexity

$f(x)$, **defined on the convex set** $S \subseteq \mathbb{R}^n$, is called μ-strongly convex (strongly convex) on S, if:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2) - \frac{\mu}{2}\lambda(1-\lambda)\|x_1 - x_2\|^2$$

for any $x_1, x_2 \in S$ and $0 \leq \lambda \leq 1$ for some $\mu > 0$.

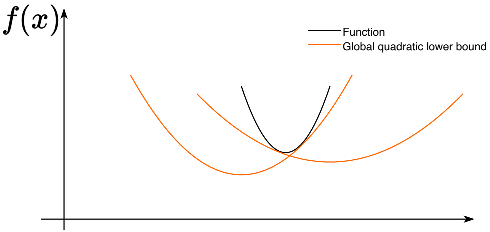


Figure 2: Strongly convex function is greater or equal than global quadratic lower bound at any point

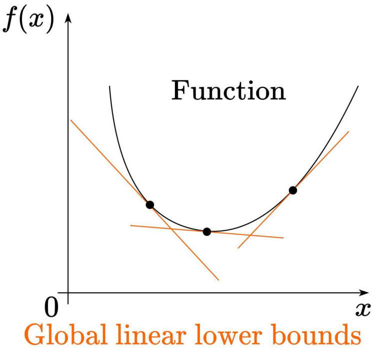
First-order differential criterion of convexity

The differentiable function $f(x)$ defined on the convex set $S \subseteq \mathbb{R}^n$ is convex if and only if $\forall x, y \in S$:

$$f(y) \geq f(x) + \nabla f^T(x)(y - x)$$

Let $y = x + \Delta x$, then the criterion will become more tractable:

$$f(x + \Delta x) \geq f(x) + \nabla f^T(x)\Delta x$$



Second-order differential criterion of strong convexity

Twice differentiable function $f(x)$ defined on the convex set $S \subseteq \mathbb{R}^n$ is μ-strongly convex if and only if $\forall x \in \mathbf{int}(S) \neq \emptyset$:

$$\nabla^2 f(x) \succeq \mu I$$

In other words:

$$\langle y, \nabla^2 f(x) y \rangle \geq \mu \|y\|^2$$

Motivational Experiment with JAX

Why convexity and strong convexity is important? Check the simple code snippet.

Problem 1

Question

Show, that $f(x) = \|x\|$ is convex on $\mathbb{R}^n$.

Question

Show, that $f(x) = x^\top A x$, where $A \succeq 0$ - is convex on $\mathbb{R}^n$.

Problem 2

Question

Show, that if $f(x)$ is convex on $\mathbb{R}^n$, then $\exp(f(x))$ is convex on $\mathbb{R}^n$.

Problem 3

Question

If $f(x)$ is convex nonnegative function and $p \geq 1$. Show that $g(x) = f(x)^p$ is convex.

Problem 4

Question

Show that, if $f(x)$ is concave positive function over convex S, then $g(x) = \frac{1}{f(x)}$ is convex.

Question

Show, that the following function is convex on the set of all positive denominators

$$f(x) = \cfrac{1}{x_1 - \cfrac{1}{x_2 - \cfrac{1}{x_3 - \cfrac{1}{\dots}}}}, x \in \mathbb{R}^n$$

Problem 5

Question

Let $S = \{x \in \mathbb{R}^n \mid x \succ 0, \|x\|_\infty \leq M\}$. Show that $f(x) = \sum_{i=1}^n x_i \log x_i$ is $\frac{1}{M}$-strongly convex.

Polyak-Lojasiewicz (PL) Condition

PL inequality holds if the following condition is satisfied for some $\mu > 0$,

$$\|\nabla f(x)\|^2 \geq \mu(f(x) - f^*) \forall x$$

The example of a function, that satisfies the PL-condition, but is not convex.

$$f(x,y) = \frac{(y - \sin x)^2}{2}$$

Example of Pl non-convex function Open in Colab.

Optimality Conditions. Important notions recap

$$f(x) \to \min_{x \in S}$$

A set S is usually called a budget set.

- A point x^* is a global minimizer if $f(x^*) \leq f(x)$ for all x.
- A point x^* is a local minimizer if there exists a neighborhood N of x^* such that $f(x^*) \leq f(x)$ for all $x \in N$.
- A point x^* is a strict local minimizer (also called a strong local minimizer) if there exists a neighborhood N of x^* such that $f(x^*) < f(x)$ for all $x \in N$ with $x \neq x^*$.
- We call x^* a stationary point (or critical) if $\nabla f(x^*) = 0$. Any local minimizer must be a stationary point.

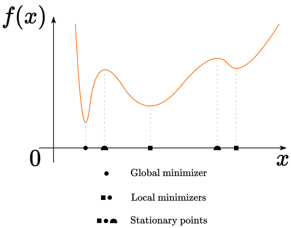


Figure 4: Illustration of different stationary (critical) points

Unconstrained optimization recap

First-Order Necessary Conditions

If x^* is a local minimizer and f is continuously differentiable in an open neighborhood, then

$$\nabla f(x^*) = 0 \tag{1}$$

Second-Order Sufficient Conditions

Suppose that $\nabla^2 f$ is continuous in an open neighborhood of x^* and that

$$\nabla f(x^*) = 0 \quad \nabla^2 f(x^*) \succ 0. \tag{2}$$

Then x^* is a strict local minimizer of f.

Lagrange multipliers recap

Consider simple yet practical case of equality constraints:

$$
\begin{split}
& f(x) \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } & h_i(x) = 0, i = 1, \ldots, p
\end{split}
$$

The basic idea of Lagrange method implies the switch from conditional to unconditional optimization through increasing the dimensionality of the problem:

$$
L(x, \nu) = f(x) + \sum\limits_{i=1}^p \nu_i h_i(x) \to \min\limits_{x \in \mathbb{R}^n, \nu \in \mathbb{R}^p}
$$

Problem 1

Question

Function $f : E \to \mathbb{R}$ is defined as

$$f(x) = \ln\left(-Q(x)\right)$$

where $E = \{x \in \mathbb{R}^n : Q(x) < 0\}$ and

$$Q(x) = \frac{1}{2}x^\top A x + b^\top x + c$$

with $A \in \mathbb{S}^n_{++}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$.
Find the maximizer x^* of the function f.

Problem 2

Question

Give an explicit solution of the following task.

$$\langle c, x \rangle + \sum_{i=1}^{n} x_i \log x_i \to \min_{x \in \mathbb{R}^n}$$

$$\text{s.t. } \sum_{i=1}^{n} x_i = 1,$$

where $x \in \mathbb{R}^n_{++}, c \neq 0$.

Adversarial Attacks as Constrained Optimization

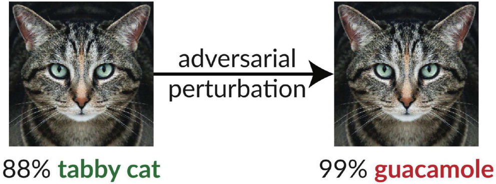


Figure 5: Any neural network can be fooled with invisible pertubation

- Targetted Adversarial Attack:

$$\rho(x, x + r) \to \min_{r \in \mathbb{R}^n}$$

$$\text{s.t. } y(x + r) = \text{target\_class},$$

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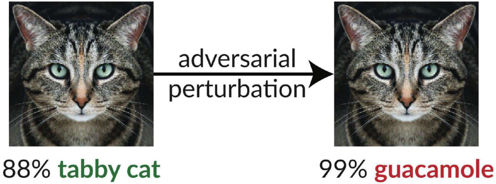

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- Non-targetted Adversarial Attack:

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Solution from Szegedy et al, “Intriguing properties of neural networks”

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! Method Problems

1. *Attack success or not* – there is no guarantee the method will work;
2. Simple optimizers may not work due to nonconvexity of Neural Networks (authors use L-BFGS);

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i More sophisticated methods

- Fast Gradient Sign Method (FGSM)
- Deep Fool